7.5 Homogeneous linear system with constant coefficients

Model Problem：

$$\dot{x} = \frac{dx}{dt} = Ax$$

where A is a real constant $n \times n$ matrix and x is a vector.

Q： How to solve $\dot{x} = Ax$ in general ?

Motivation：

For $n = 1$, the system reduces to a single first order equation $\dot{x} = ax$. And the solution can be easily written as $x = ce^{at}$.

Note that $x \equiv 0$ is a solution.

- If $a < 0$, then $x = ce^{at}$, $\therefore$ all other solutions $\to 0$ as $t \to \infty$. In this case, we say that $x = 0$ is an **asymptotically stable** equilibrium solution.
- If $a > 0$, then all other solutions depart from $x = 0$ as t increases. In this case, $x = 0$ is unstable.

Q： How to solve $\dot{x} = Ax$ for $n > 1$?

We begin with a simple example.

ex： Find the general solution of the system $\dot{x}=\begin{pmatrix}2 & 0\\ 0 & -3\end{pmatrix}x$.

sol： Let $x=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$, then $\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \Rightarrow \begin{matrix}\dot{x}_1= 2x_1\\ \dot{x}_2= -3x_2\end{matrix}$

It's a decoupled system of equations, so we can solve these two equations separately.

$\therefore\ x_1=c_1e^{2t},\ x_2=c_2e^{-3t} \Rightarrow x=\begin{pmatrix}c_1e^{2t}\\ c_2e^{-3t}\end{pmatrix}=c_1\begin{pmatrix}e^{2t}\\ 0\end{pmatrix}+c_2\begin{pmatrix}0\\ e^{-3t}\end{pmatrix}$

$\therefore$ Let $x^{(1)}(\mathrm{t})=\begin{pmatrix}e^{2t}\\ 0\end{pmatrix},\ x^{(2)}(\mathrm{t})=\begin{pmatrix}0\\ e^{-3t}\end{pmatrix} \Rightarrow W(x^{(1)},x^{(2)})=\begin{vmatrix}e^{2t} & 0\\ 0 & e^{-3t}\end{vmatrix}=e^{-t}\neq 0$

$\therefore\ x^{(1)}(\mathrm{t})$ & $x^{(2)}(\mathrm{t})$ form a fundamental set of solutions, and the general solution is given by $x(\mathrm{t})=c_1x^{(1)}(\mathrm{t})+c_2x^{(2)}(\mathrm{t})$

Phase Plane

- When $n = 2$, then the system reduces to

$$x_1' = a_{11}x_1 + a_{12}x_2$$
$$x_2' = a_{21}x_1 + a_{22}x_2$$

- This case can be visualized in the x_1x_2-plane, which is called the **phase plane**.
- In the phase plane, a direction field can be obtained by evaluating **Ax** at many points and plotting the resulting vectors, which will be tangent to solution vectors.
- A plot that shows representative solution trajectories is called a **phase portrait**.
- Examples of phase planes, directions fields, and phase portraits will be given later in this section.

For the general system, we guess the solution is of the form $x = \xi e^{rt}$, where ξ and r are yet to be determined.

If $x = \xi e^{rt}$ is a vector solution to $\dot{x} = \mathrm{A}x$, then we have $r\xi e^{rt} = \mathrm{A}\xi e^{rt} \Rightarrow \quad \mathrm{A}\xi = r\xi$.

$\therefore$ r is an eigenvalue of A and ξ is the associated eigenvector.

$\Rightarrow \quad (\mathrm{A} - \mathrm{rI})\xi = 0 \quad \therefore \quad \det(\mathrm{A} - \mathrm{rI}) = 0$.

So what's the moral of the story？

If we can find the eigenvalues, then we can write down the solutions ！

We begin with case of 2×2 coefficient matrices, the general $n \times n$ system will to discussed later on.

case 1： Distinct eigenvalues

case 2： Repeated eigenvalues

case 3： Complex eigenvalues

case 1： Distinct eigenvalues

Ex： Find the solution of the following system $\dot{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$

sol： as we discussed before, we try to find the eigenvalues.

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0 \quad \Rightarrow \quad 1-\lambda = \pm 2$$

$\therefore \ \lambda = -1 \ \ or \ \ 3$, to find the solution.

We need to find the eigenvectors.

$$A\xi = -\xi \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = -\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

$$\therefore \quad \xi_1 + \xi_2 = -\xi_1 \quad \Rightarrow \quad 2\xi_1 = -\xi_2$$

$$\therefore \quad \xi = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$A\eta = 3\eta \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 3\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$\therefore \quad \eta_1 + \eta_2 = 3\eta_1 \quad \Rightarrow \quad \eta_2 = 2\eta_1$$

$$\therefore \quad \eta = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore$ We have found two solutions

$$(\xi e^{-t}, \eta e^{3t}) = \begin{pmatrix} e^{-t} & , & e^{3t} \\ -2e^{-t} & , & 2e^{3t} \end{pmatrix} \quad \Rightarrow \quad \mathrm{d\,e\,t} \neq$$

By Theorem 7.4.2, we know that any solution of $\dot{x} = Ax$ can be written as

$$x = \begin{pmatrix} c_2 e^{-t} \\ -2c_2 e^{-t} \end{pmatrix} + \begin{pmatrix} c_1 e^{3t} \\ 2c_1 e^{3t} \end{pmatrix} = \begin{pmatrix} c_2 e^{-t} + c_1 e^{3t} \\ -2c_2 e^{-t} + 2c_1 e^{3t} \end{pmatrix}$$

$$= c_2 \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + c_1 \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} = c_1 x^{(1)}(\mathrm{t}) + c_2 x^{(2)}(\mathrm{t})$$

$$W(x^{(1)}, x^{(2)}) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -2e^{2t} - 2e^{2t} = -4e^{2t} \neq 0$$

$\therefore$ $x^{(1)}$ and $x^{(2)}$ from a fundamental set of solutions.

The case $n = 2$ is special, if we let

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 4x_1 + x_2 \end{pmatrix}$$

There are two ways to visualize the solutions

1： To visualize solutions in $x_1 x_2$ - plane,

this is so called the phase plane

2： To visualize solutions as functions of t.

If we want to visualize the behavior of solutions in $x_1 x_2$ - plane,

$$x_1 = c_1 e^{3t} + c_2 e^{-t} \qquad x_2 = 2c_1 e^{3t} - 2c_2 e^{-t}.$$

How to get a relation between x_1 and x_2 ?

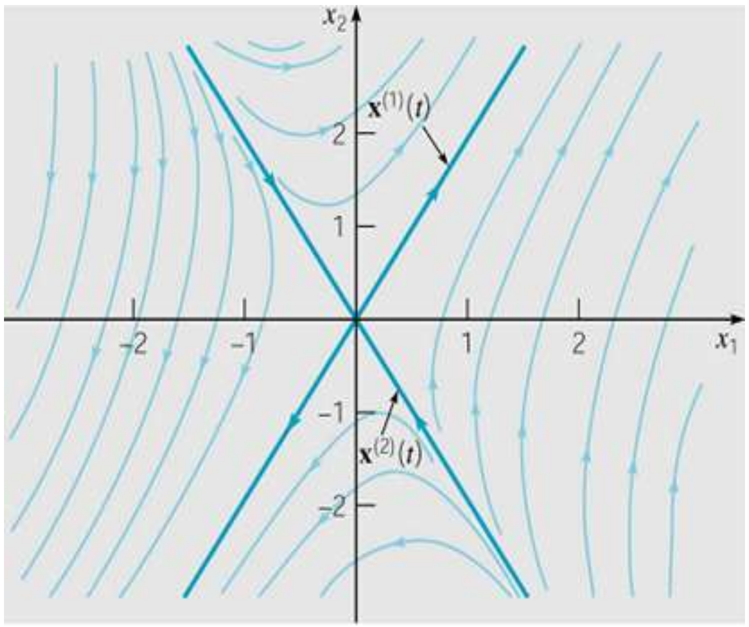
$$2x_1 + x_2 = 4c_1 e^{3t}, \quad -2x_1 + x_2 = -4c_2 e^{-t}$$

$$\therefore \quad (2x_1 + x_2)(-2x_1 + x_2)^3 = \text{constant} = c$$

We draw the curves of x_1, x_2 according to different values of c.

When $c = 0$, we have two straight lines

$$2x_1 + x_2 = 0, \quad -2x_1 + x_2 = 0$$



$$(2x_1 + x_2)(-2x_1 + x_2)^3 = c$$

$$\dot{x}_1 = x_1 + x$$
$$\dot{x}_2 = 4x_1 + x$$

To determine how x_1, x_2 behave as t increases,
we look at the solution formula

$$x_1 = c_1 e^{3t} + c_2 e^{-t}, \quad x_2 = 2c_1 e^{3t} - 2c_2 e^{-t}$$

- For the points on $2x_1 + x_2 = 0$, $\begin{cases} x_1 = c_2 e^{-t} \\ x_2 = -2c_2 e^{-t} \end{cases}$

  then $(x_1, x_2) \to 0$ as $t \to \infty$.

- For the points on $-2x_1 + x_2 = 0$, $\begin{cases} x_1 = c_1 e^{3t} \\ x_2 = 2c_1 e^{3t} \end{cases}$

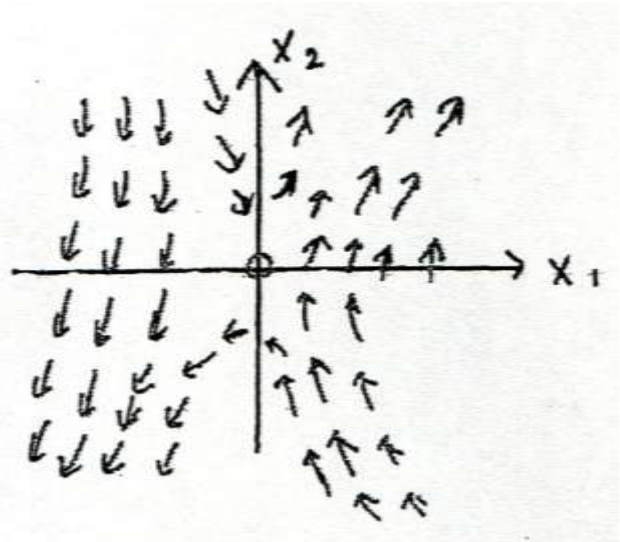
  then $(x_1, x_2) \to \pm\infty$ as $t \to \infty$.

Then we can draw the phase space diagram.

Remark：

To draw the phase space diagram, we need to solve the problem first. Without solving the problem we can also draw a rough picture by using the concept of direction field.

$$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = 4x_1 + x \end{cases}$$



In each point (x_1, x_2) in the phase space, we associate it with a unit tangent vector in the direction of $(x_1 + x_2, 4x_1 + x_2)$.

$(1,1) \rightarrow (2,5)$

$(1,0) \rightarrow (1,4)$

$\vdots$

$$\frac{dx_2}{dx_1} = \frac{4x_1 + x_2}{x_1 + x_2}$$

The direction field describes the rough behavior of solutions.

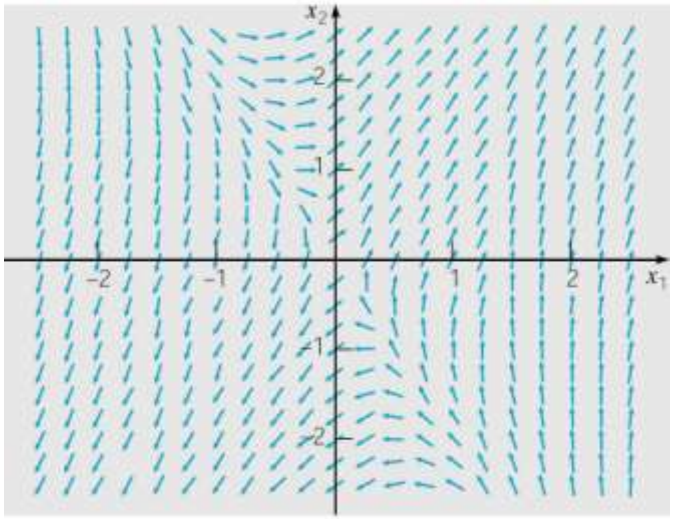
Example 2: Direction Field (1 of 9)

- Consider the homogeneous equation **x**' = **Ax** below.

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x}$$

- A direction field for this system is given below.
- Substituting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ in for **x**, and rewriting system as $(\mathbf{A}\text{-}r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, we obtain

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



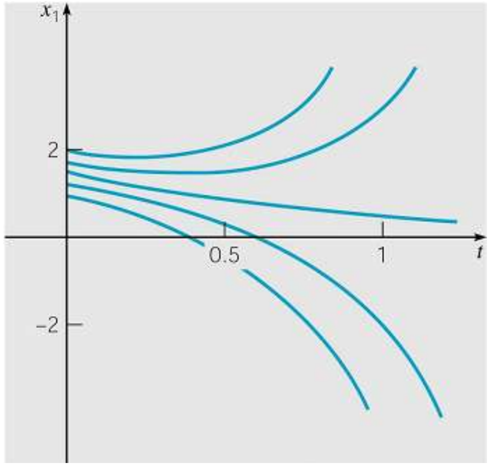
Example 2: Time Plots for General Solution (9 of 9)

- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}(t) = c_1\begin{pmatrix}1\\2\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-2\end{pmatrix}e^{-t} \quad \Leftrightarrow \quad \begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = \begin{pmatrix}c_1e^{3t} + c_2e^{-t}\\2c_1e^{3t} - 2c_2e^{-t}\end{pmatrix}$$

- As an alternative to phase plane plots, we can graph x_1 or x_2 as a function of t. A few plots of x_1 are given below.
- Note that when $c_1 = 0$, $x_1(t) = c_2e^{-t} \to 0$ as $t \to \infty$.

  Otherwise, $x_1(t) = c_1e^{3t} + c_2e^{-t}$ grows unbounded as $t \to \infty$.
- Graphs of x_2 are similarly obtained.



Example 3: Direction Field (1 of 9)

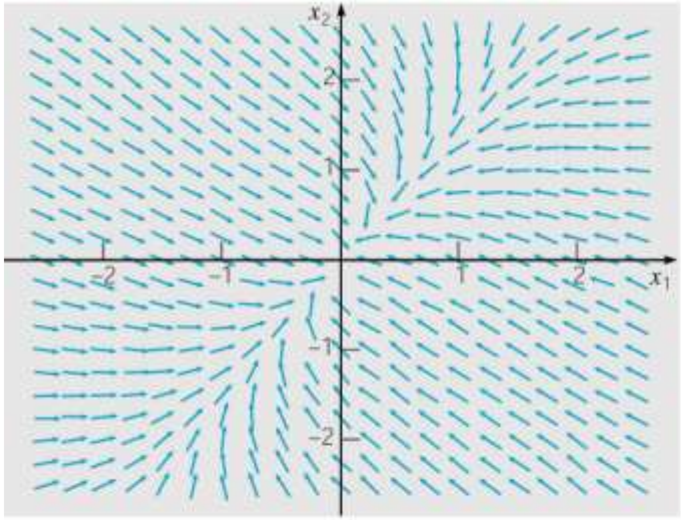
- Consider the homogeneous equation **x'** = **Ax** below.

$$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x} \qquad \begin{aligned} \dot{x}_1 &= -3x_1 + \sqrt{2}x_2 \\ \dot{x}_2 &= \sqrt{2}x_1 - 2x_2 \end{aligned}$$

- A direction field for this system is given below.
- Substituting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ in for **x**, and rewriting system as

$(\mathbf{A}-r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, we obtain

$$\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Example 3: Eigenvalues (2 of 9)

- Our solution has the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, where r and $\boldsymbol{\xi}$ are found by solving

$$\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Recalling that this is an eigenvalue problem, we determine r by solving det($\mathbf{A}$-$r\mathbf{I}$) = 0:

$$\begin{vmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{vmatrix} = (-3-r)(-2-r)-2 = r^2+5r+4 = (r+1)(r+4)$$

- Thus r_1 = -1 and r_2 = -4.

Example 3: First Eigenvector (3 of 9)

- Eigenvector for r_1 = -1: Solve

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \Leftrightarrow \begin{pmatrix} -3+1 & \sqrt{2} \\ \sqrt{2} & -2+1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ \sqrt{2} & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \sqrt{2}/2\xi_2 \\ \xi_2 \end{pmatrix} \to \text{choose } \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

Example 3: Second Eigenvector (4 of 9)

- Eigenvector for r_2 = -4: Solve

$$(\mathbf{A}-r\mathbf{I})\boldsymbol{\xi}=\mathbf{0} \Leftrightarrow \begin{pmatrix} -3+4 & \sqrt{2} \\ \sqrt{2} & -2+4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \boldsymbol{\xi}^{(2)}=\begin{pmatrix} -\sqrt{2}\xi_2 \\ \xi_2 \end{pmatrix}$$

$$\to \text{choose } \boldsymbol{\xi}^{(2)}=\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

Example 3: General Solution (5 of 9)

- The corresponding solutions $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

- The Wronskian of these two solutions is

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right](t) = \begin{vmatrix} e^{-t} & -\sqrt{2}e^{-4t} \\ \sqrt{2}e^{-t} & e^{-4t} \end{vmatrix} = 3e^{-5t} \neq 0$$

- Thus $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are fundamental solutions, and the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) \\ &= c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} \end{aligned}$$

Example 3: Phase Plane for $\mathbf{x}^{(1)}$ (6 of 9)

- To visualize solution, consider first $\mathbf{x} = c_1\mathbf{x}^{(1)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}e^{-t} \quad \Leftrightarrow \quad x_1 = c_1 e^{-t},\ \ x_2 = \sqrt{2}c_1 e^{-t}$$

- Now

$$x_1 = c_1 e^{-t},\ x_2 = \sqrt{2}c_1 e^{-t} \ \Leftrightarrow \ e^{-t} = \frac{x_1}{c_1} = \frac{x_2}{\sqrt{2}c_1} \ \Leftrightarrow \ x_2 = \sqrt{2}x_1$$

- Thus $\mathbf{x}^{(1)}$ lies along the straight line $x_2 = 2^{1/2} x_1$, which is the line through origin in direction of first eigenvector $\boldsymbol{\xi}^{(1)}$
- If solution is trajectory of particle, with position given by (x_1, x_2), then it is in Q1 when $c_1 > 0$, and in Q3 when $c_1 < 0$.
- In either case, particle moves towards origin as t increases.

Example 3: Phase Plane for $\mathbf{x}^{(2)}$ (7 of 9)

- Next, consider $\mathbf{x} = c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} \quad \Leftrightarrow \quad x_1 = -\sqrt{2}c_2 e^{-4t}, \ \ x_2 = c_2 e^{-4t}$$

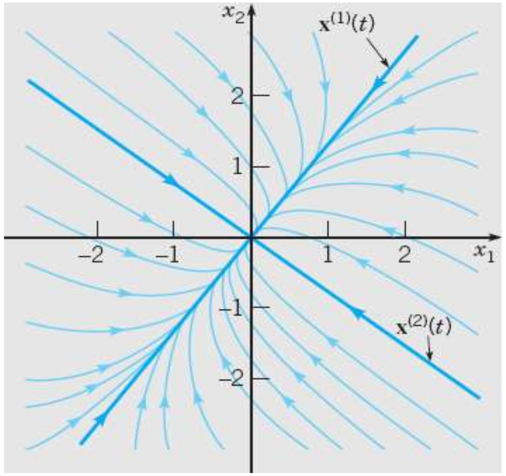
- Then $\mathbf{x}^{(2)}$ lies along the straight line $x_2 = -2^{½} x_1$, which is the line through origin in direction of 2nd eigenvector $\boldsymbol{\xi}^{(2)}$
- If solution is trajectory of particle, with position given by (x_1, x_2), then it is in Q4 when $c_2 > 0$, and in Q2 when $c_2 < 0$.
- In either case, particle moves towards origin as t increases.

Example 3:
Phase Plane for General Solution (8 of 9)

- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

- As $t \to \infty$, $c_1\mathbf{x}^{(1)}$ is dominant and $c_2\mathbf{x}^{(2)}$ becomes negligible. Thus, for $c_1 \neq 0$, all solutions asymptotically approach origin along the line $x_2 = 2^{1/2} x_1$ as $t \to \infty$.
- Similarly, all solutions are unbounded as $t \to -\infty$.
- The origin is a **node**, and is asymptotically stable.

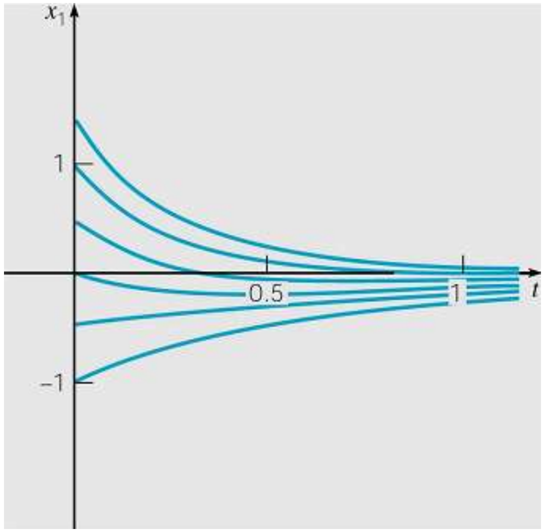


Example 3:
Time Plots for General Solution (9 of 9)

- The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$:

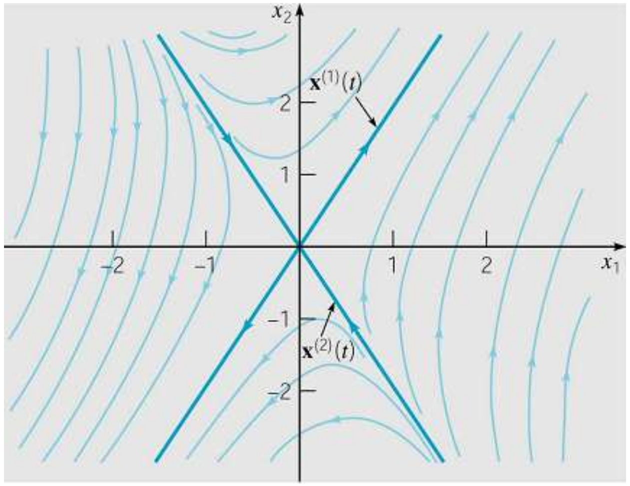
$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} \iff \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} - \sqrt{2}c_2 e^{-4t} \\ \sqrt{2}c_1 e^{-t} + c_2 e^{-4t} \end{pmatrix}$$

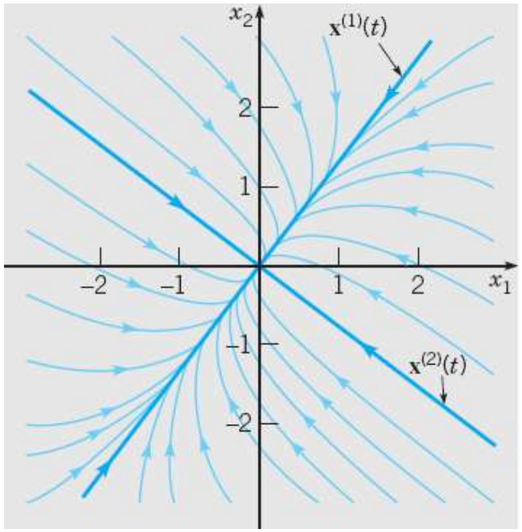
- As an alternative to phase plane plots, we can graph x_1 or x_2 as a function of t. A few plots of x_1 are given below.
- Graphs of x_2 are similarly obtained.



2 x 2 Case:
Real Eigenvalues, Saddle Points and Nodes

- The previous two examples demonstrate the two main cases for a 2 x 2 real system with real and different eigenvalues:
  - Both eigenvalues have opposite signs, in which case origin is a saddle point and is unstable.
  - Both eigenvalues have the same sign, in which case origin is a node, and is asymptotically stable if the eigenvalues are negative and unstable if the eigenvalues are positive.





We conclude this section with the general system

$$\dot{x} = Ax \qquad A：n \times n \text{ matrix}$$

So far we are still in the case of n distinct and real eigenvalues.

So to find the solutions, we proceed as before：

1) Find the eigenvalues $r_1, \cdots, r_n$ by solving the algebraic equation

$$\det(A - rI) = 0 .$$

2) Find the eigenvectors $\xi^{(1)}, \cdots, \xi^{(n)}$ associated with $r_1, r_2, \cdots, r_n$.

Note that $\xi^{(1)}, \cdots, \xi^{(n)}$ are linearly independent.

$\therefore$ The corresponding solutions of the system are

$$x^{(1)}(t) = \xi^{(1)} e^{r_1 t}, \cdots, x^{(n)}(t) = \xi^{(n)} e^{r_n t}.$$

To show that $x^{(1)}(t),\cdots,x^{(n)}(t)$ for a fundamental set of solutions, we compute their Wronskian：

$$W\left[x^{(1)},\cdots,x^{(n)}\right](t)=\begin{vmatrix}\xi_1^{(1)}e^{r_1t} & \cdots & \xi_1^{(n)}e^{r_nt}\\ \vdots & & \vdots\\ \xi_n^{(1)}e^{r_1t} & \cdots & \xi_n^{(n)}e^{r_nt}\end{vmatrix}$$

$$=e^{(r_1+\cdots+r_n)t}\underbrace{\begin{vmatrix}\xi_1^{(1)} & \cdots & \xi_1^{(n)}\\ \vdots & & \vdots\\ \xi_n^{(1)} & \cdots & \xi_n^{(n)}\end{vmatrix}}_{\neq 0\ (\because\ \text{independent})}$$

3) Now we can form the general solution of the system

$$x = c_1 \xi^{(1)} e^{r_1 t} + \cdots + c_n \xi^{(n)} e^{r_n t}.$$

Remark：

If **A** is real and symmetric, then all eigenvalues are real.
Even if some of the eigenvalues are repeated, there is still a full set of n eigenvectors $\xi^{(1)}, \cdots, \xi^{(n)}$.
So the solution formula is still valid.

Ex： Find the general solution of $\dot{x} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} x$

sol：

$$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = 0$$

$$= -\lambda^3 + 1 + 1 + \lambda + \lambda + \lambda = -\lambda^3 + 3\lambda + 2$$

$$= -(\lambda^3 - 3\lambda - 2) = -(\lambda+1)(\lambda^2 - \lambda - 2)$$

$$= -(\lambda+1)(\lambda+1)(\lambda-2) = -(\lambda+1)^2(\lambda-2)$$

$$\therefore \quad \lambda = -1, -1, 2$$

To find $\xi^{(1)}$, we solve $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \\ \xi_3^{(1)} \end{bmatrix} = 0$

$\therefore \quad \xi_1^{(1)} + \xi_2^{(1)} + \xi_3^{(1)} = 0$

$\therefore$ We can choose $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

For $\lambda = 2$, we have $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} \xi_1^{(3)} \\ \xi_2^{(3)} \\ \xi_3^{(3)} \end{bmatrix} = 0$

$\therefore \quad -2\xi_1^{(3)} + \xi_2^{(3)} + \xi_3^{(3)} = 0 \quad \Rightarrow \quad \xi^{(3)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ The general solution can be written as

$$x(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}$$