7.8 Repeated eigenvalues

We begin with an example

$$\dot{x} = \mathrm{A}x = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} x$$

If we solve for the eigenvalues and eigenvectors.

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = 0 \qquad \lambda = 2$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$v_1 - v_2 = 2v_1 \qquad \Rightarrow \quad v_1 = -v_2 \qquad \therefore \quad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \text{ one solution is } \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$$

We seek for the second solution of the form

$$x(t) = \xi t e^{2t} + \eta e^{2t} \qquad (*)$$

(If we only assume the first one, we find that $\xi = 0$.)

If we play $(*)$ into the ODE system, we obtain

$$\dot{x} = \xi e^{2t} + 2\xi t e^{2t} + 2\eta e^{2t} = Ax = A\xi t e^{2t} + A\eta e^{2t}$$

$$\Rightarrow \begin{cases} \xi + 2\eta = A\eta & \Rightarrow (A - 2I)\eta \\ A\xi = 2\xi & \Rightarrow (A - 2I)\xi = 0 \end{cases}$$

$\therefore\ \xi$ is the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ associated

with the eigenvalue 2.

η is called the generalized eigenvector

corresponding to the eigenvalue 2.

To solve η from the equation

$$(A-2I)\eta=\xi \qquad (A-2I)^2\eta =$$

$$\begin{pmatrix} 1-2 & -1 \\ 1 & 3-2 \end{pmatrix}\eta=\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \ -\eta_1-\eta_2=1 \quad \Rightarrow \quad \eta_2=-1 \ , \ \eta_1=0$$

So the second solution can be written as

$$x_2(t)=\xi te^{2t}+\eta e^{2t}=\begin{pmatrix} te^{2t} \\ -te^{2t}-e^{2t} \end{pmatrix}$$

$\therefore$ The general solution is

$$x(t)=c_1\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}+c_2\begin{pmatrix} te^{2t} \\ -te^{2t}-e^{2t} \end{pmatrix}$$

Q： What is the fundamental matrix in this case？

$$\psi(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{pmatrix}$$

＊ Another way to find $\psi(t)$ ：

If we look at the Jordon form of $\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} = \mathrm{A}$

Recall that we found $\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \mathrm{A}\xi = 2\xi$

$$\eta = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \mathrm{A}\ \eta = 2\ \eta +$$

$$\therefore\ \mathrm{A}\begin{bmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{bmatrix}}_{\mathrm{T}} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}_{J} \qquad \therefore\ \mathrm{A} = \mathrm{T}J\mathrm{T}^{-1}$$

J is the A's Jordon form.

$$A=\begin{bmatrix}1 & -1\\ 1 & 3\end{bmatrix}=TJT^{-1}=\begin{bmatrix}1 & 0\\ -1 & -1\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & -1\end{bmatrix}$$

$\therefore$ By letting $T^{-1}x=y \quad \Rightarrow \quad \dot{x}=Ax=TJT^{-1}x$

$$\Rightarrow \quad T^{-1}\dot{x}=JT^{-1}x \quad \Rightarrow \quad \dot{y}=Jy$$

$$\therefore \quad \begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} \qquad \begin{cases}\dot{y}_1=2y_1+y\\ \dot{y}_2=2y_2\end{cases}$$

$$\Rightarrow \quad y_2=c_2e^{2t}, \quad \dot{y}_1=2y_1+c_2e^{2t} \quad \Rightarrow \quad \dot{y}_1-2y_1-c_2e^{2t}=0$$

$$\Rightarrow \quad \dot{y}_1e^{-2t}-2y_1e^{-2t}-c_2=0 \qquad \therefore \quad \left(y_1e^{-2t}\right)'=c_2$$

$$\therefore \quad y_1e^{-2t}=c_2t+d \quad \Rightarrow \quad y_1=c_2te^{2t}+de^{2t}$$

$$\therefore \quad y_1=c_1e^{2t}+c_2te^{2t}$$

$$\dot{y}_1=2c_1e^{2t}+c_2e^{2t}+2c_2te^{2t}=2y_1+c_2e^{2t}$$

$$\therefore \quad \dot{y} = Jy \text{ has solution } \begin{pmatrix} c_1 e^{2t} + c_2 t e^{2t} \\ c_2 e^{2t} \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} te^{2t} \\ e^{2t} \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} te^{2t} \\ e^{2t} \end{pmatrix} \text{ are two fundamental solutions.}$$

To find fundamental solutions for $\dot{x} = \mathrm{A}x$, we recall the transformation $y = \mathrm{T}^{-1}x \;\Rightarrow\; x = \mathrm{T}y$

$$\therefore \quad x^{(1)}(t) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$$

$$\Rightarrow \quad x^{(2)}(t) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} te^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} te^{2t} \\ -te^{2t} - e^{2t} \end{pmatrix}$$

$$\therefore \quad \psi(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{pmatrix}$$

ex： $\dot{x} = \mathrm{A}x = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} x$

To compute $\Phi(t)$ satisfying $\begin{cases} \dot{\Phi}(t) = \mathrm{A}\Phi \\ \Phi(0) = \mathrm{I} \end{cases}$, we have three methods.

Method 1 ： Solving linear equations.

We have already found a fundamental matrix $\psi(t)$,

where $\psi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}$

$$x^{(1)} = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} te^{2t} \\ -te^{2t} - e^{2t} \end{pmatrix}$$

$\therefore$ Let $\Phi(t) = \begin{bmatrix} c_1 x^{(1)} + c_2 x^{(2)} & c_3 x^{(1)} + c_4 x^{(2)} \end{bmatrix}$

$$= \begin{bmatrix} c_1 e^{2t} + c_2 te^{2t} & c_3 e^{2t} + c_4 te^{2t} \\ -c_1 e^{2t} - c_2 te^{2t} - c_2 e^{2t} & -c_3 e^{2t} - c_4 te^{2t} - c_4 e^{2t} \end{bmatrix}$$

From the initial condition $\Phi(0) = \mathrm{I}$.

$$\Rightarrow \begin{bmatrix} c_1 & c_3 \\ -c_1 - c_2 & -c_3 - c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad c_1 = 1, \quad c_2 = -1, \quad c_3 = 0, \quad c_4 = -1$$

$$\therefore \quad \Phi(t) = \begin{bmatrix} e^{2t} - te^{2t} & -te^{2t} \\ -e^{2t} + te^{2t} + e^{2t} & te^{2t} + e^{2t} \end{bmatrix} = \begin{bmatrix} e^{2t} - te^{2t} & -te^{2t} \\ te^{2t} & te^{2t} + e^{2t} \end{bmatrix}$$

Method 2 ： From the formula $\Phi(t) = \psi(t)\psi(0)^{-1}$

$$= \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} - te^{2t} & -te^{2t} \\ te^{2t} & te^{2t} + e^{2t} \end{bmatrix}$$

Method 3 ：

Now we try to find e^{At} from the formula itself

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots + \frac{(At)^n}{n!} + \cdots, \text{ where } A = TJT^{-1}$$

$$\therefore \quad e^{At} = I + TJT^{-1}t + \frac{TJ^2T^{-1}}{2!}t^2 + \cdots + \frac{TJ^nT^{-1}}{n!}t^n + \cdots$$

$$= T(I + Jt + \frac{(Jt)^2}{2!} + \cdots + \frac{(Jt)^n}{n!} + \cdots)T^{-1}$$

$\therefore$ we need to compute $(Jt)^n$.

$$Jt = \begin{bmatrix} 2t & t \\ 0 & 2t \end{bmatrix}$$

$$(Jt)^2 = \begin{bmatrix} 2t & t \\ 0 & 2t \end{bmatrix}\begin{bmatrix} 2t & t \\ 0 & 2t \end{bmatrix} = \begin{bmatrix} 4t^2 & 2t^2 + 2t^2 \\ 0 & 4t^2 \end{bmatrix} = \begin{bmatrix} 4t^2 & 4t^2 \\ 0 & 4t^2 \end{bmatrix}$$

$$(Jt)^3 = \begin{bmatrix} 4t^2 & 4t^2 \\ 0 & 4t^2 \end{bmatrix}\begin{bmatrix} 2t & t \\ 0 & 2t \end{bmatrix} = \begin{bmatrix} 8t^3 & 12t^3 \\ 0 & 8t^3 \end{bmatrix}$$

$$(Jt)^4 = \begin{bmatrix} 8t^3 & 12t^3 \\ 0 & 8t^3 \end{bmatrix}\begin{bmatrix} 2t & t \\ 0 & 2t \end{bmatrix} = \begin{bmatrix} 16t^4 & 32t^4 \\ 0 & 16t^4 \end{bmatrix}$$

The pattern for the diagonal component is clear.
We need to find the one on top.

$$0 + t + \frac{1}{2!}4t^2 + \frac{1}{3!}12t^3 + \frac{1}{4!}32t^4 + \cdots$$

$$= t(1 + 2t + \frac{1}{2!}(2t)^2 + \frac{1}{3!}(2t)^3 + \cdots) = te^{2t}$$

$$\therefore \quad e^{\mathrm{A}t} = \mathrm{T}\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\mathrm{T}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -te^{2t} - e^{2t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} e^{2t} - te^{2t} & -te^{2t} \\ te^{2t} & te^{2t} + e^{2t} \end{bmatrix} = \Phi(t)$$

Lemma： Let $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \Rightarrow \quad J^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$

(By induction)

7.9 Non-homogeneous linear system

$$\dot{x} = p(t)x + g(t)$$

To solve the nonhomogeneous problem, we have three ways：

1) undetermined coefficients
2) Diagonalization
3) Variation of parameters
4) Laplace transform

Before, we talked about (3) by using the fundamental matrix.
Now we will begin with (1).

1) **undetermined coefficients**

Consider $\dot{x} = \mathrm{A}x + g(t)$

This method is used only if A is a constant matrix, and if the components of g are polynomial, exponential, or trigonometric functions or sums or products of these.

Ex： $\dot{x} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$

Before we already solved the homogeneous problem：

the eigenvalues are -3 and -1, and the corresponding

eigenvectors $\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\therefore$ The general solution of the homogeneous system is

$$x = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

Q： How do we guess the solution of the nonhomogeneous system？

To guess a special solution, we look at the nonhomogeneous term $\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$.

Notice that -1 is an eigenvalue of the matrix A,

∴ like the scalar equation we studied before, we guess the special solution is of the form $ate^{-t} + be^{-t} + ct + d = x(t)$ where a, b, c, d are vectors.

Now we need to determine a, b, c, d from the system

$$\dot{x}(t) = ae^{-t} - ate^{-t} - be^{-t} + c = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1e^{-t} - a_1te^{-t} - b_1e^{-t} + c_1 \\ a_2e^{-t} - a_2te^{-t} - b_2e^{-t} + c_2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a_1te^{-t} + b_1e^{-t} + c_1t + d_1 \\ a_2te^{-t} + b_2e^{-t} + c_2t + d_2 \end{pmatrix} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

$$= \begin{pmatrix} -2a_1te^{-t} - 2b_1e^{-t} - 2c_1t - 2d_1 + a_2te^{-t} + b_2e^{-t} + c_2t + d_2 \\ a_1te^{-t} + b_1e^{-t} + c_1t + d_1 - 2a_2te^{-t} - 2b_2e^{-t} - 2c_2t - 2d_2 \end{pmatrix} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

To match the coefficients, we have

$$(te^{-t}):\quad -a_1 = -2a_1 + a_2 \quad \Rightarrow \quad a_1 = a_2$$

$$(e^{-t}):\quad a_1 - b_1 = -2b_1 + b_2 + 2 \quad \Rightarrow \quad a_1 + b_1 - b_2 = 2$$

$$a_2 - b_2 = b_1 - 2b_2 \Rightarrow \quad a_2 - b_1 + b_2 = 0$$

$$\Rightarrow a_1 + a_2 = 2 \qquad \therefore \quad a_1 = a_2 = 1$$

$$\Rightarrow \quad 2b_1 - 2b_2 = 2 \quad \Rightarrow b_1 - b_2 \qquad \therefore \quad b_1 = b_2 + 1$$

$$(t):\quad -2c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = 2c_1$$

$$c_1 - 2c_2 + 3 = 0 \qquad \therefore \quad c_1 = 1, \quad c_2 = 2$$

$$(\text{constant}):\quad c_1 = -2d_1 + d_2 \quad \Rightarrow \quad 1 = -2d_1 + d_2$$

$$c_2 = d_1 - 2d_2 \quad \Rightarrow \quad 2 = d_1 - 2$$

$$\Rightarrow \quad 5 = -3d_2 \qquad \therefore d_2 = -\frac{5}{3}, \quad d_1 = -\frac{4}{3}$$

$$\therefore \quad a=\begin{pmatrix}1\\1\end{pmatrix},\ \ b=\begin{pmatrix}k+1\\k\end{pmatrix},\ \ c=\begin{pmatrix}1\\2\end{pmatrix},\ \ d=\begin{pmatrix}-\frac{4}{3}\\-\frac{5}{3}\end{pmatrix}$$

$\therefore$ The special solution we found is $\begin{pmatrix}1\\1\end{pmatrix}te^{-t}+\begin{pmatrix}k+1\\k\end{pmatrix}e^{-t}+\begin{pmatrix}1\\2\end{pmatrix}t+\begin{pmatrix}-\frac{4}{3}\\-\frac{5}{3}\end{pmatrix}$

$\therefore$ We can choose $k=0$ and get the special solution as

$$\begin{pmatrix}1\\1\end{pmatrix}te^{-t}+\begin{pmatrix}1\\0\end{pmatrix}e^{-t}+\begin{pmatrix}1\\2\end{pmatrix}t+\begin{pmatrix}-\frac{4}{3}\\-\frac{5}{3}\end{pmatrix}$$

Then the general solution of nonhomogeneous system is the sum of homogeneous solution and the particular solution we have just found.

2) **Diagonalization**

To use the method of diagonalization, we proceed as follows：

$\dot{x} = \mathrm{A}x + g(t)$

And A is diagonalizable with the Jordon form $\mathrm{A}=\mathrm{TDT}^{-1}$.

Then we have $\dot{x} = \mathrm{TDT}^{-1}x + g(t)$.

If we apply T^{-1} on both sides of the above equation and obtain

$(\mathrm{T}^{-1}x)' = \mathrm{DT}^{-1}x + \mathrm{T}^{-1}g(t) \equiv \mathrm{DT}^{-1}x + h(t)$.

Now let $\mathrm{T}^{-1}x = y$, then the equation for y can be written as

$$\dot{y} = \mathrm{D}y + h(t) \quad \text{, where} \quad \mathrm{D} = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}, \quad h = \begin{pmatrix} h_1(t) \\ \vdots \\ h_n(t) \end{pmatrix}$$

$$\therefore \quad y_j(t)' = d_j y_j(t) + h_j(t) \quad \Rightarrow \quad y_j(t)' - d_j y_j(t) = h_j(t)$$

$$\therefore \quad e^{-d_j t} y_j(t)' - d_j e^{-d_j t} y_j(t) = e^{-d_j t} h_j(t)$$

$$\Rightarrow \quad \frac{d}{dt}\left[e^{-d_j t} y_j(t)\right] = e^{-d_j t} h_j(t)$$

$$\Rightarrow \quad e^{-d_j t} y_j(t) = e^{-d_j t_0} y_j(t_0) + \int_{t_0}^{t} e^{-d_j s} h_j(s)\,ds$$

$$\Rightarrow \quad y_j(t) = y_j(t_0) + e^{d_j t}\int_{t_0}^{t} e^{-d_j s} h_j(s)\,ds\,, \quad j = 1, 2, \cdots, n$$

From $x = \mathrm{T}y$, we know the solution x.

Ex： $\dot{x} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathrm{A}x + g(t)$

eigenvalues： -3, -1

eigenvectors： $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The general solution of the homogeneous system is

$$x(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

$$\mathrm{A=TDT^{-1}} \quad \Rightarrow \quad \mathrm{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \mathrm{T}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\dot{x} = \mathrm{TDT}^{-1}x + g(t)$$

Let $\mathrm{T}^{-1}x = y \quad \Rightarrow \quad (\mathrm{T}^{-1}x)' = \mathrm{DT}^{-1}x + \mathrm{T}^{-1}g(t)$

$$\Rightarrow \quad \dot{y} = \mathrm{D}y + \mathrm{T}^{-1}g(t) = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} y + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

$$\Rightarrow \quad \begin{cases} \dot{y}_1 + 3y_1 = \sqrt{2}e^{-t} - \dfrac{3}{\sqrt{2}}t \\ \dot{y}_2 + y_2 = \sqrt{2}e^{-t} + \dfrac{3}{\sqrt{2}}t \end{cases} \quad \rightarrow \text{ decoupled system}$$

$\therefore$ These two equations can be solved by the method of integrating factors.

$$\Rightarrow \begin{cases} y_1 = \dfrac{\sqrt{2}}{2}e^{-t} - \dfrac{3}{\sqrt{2}}\left[\left(\dfrac{t}{3}\right) - \dfrac{1}{9}\right] + c_1 e^{-3t} \\ \\ y_2 = \sqrt{2}te^{-t} + \dfrac{3}{\sqrt{2}}(t-1) + c_2 e^{-t} \end{cases}$$

$$\therefore \quad x = \mathrm{T}y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\dfrac{c_1}{\sqrt{2}}\right)e^{-3t} + \left[\dfrac{c_2}{\sqrt{2}} + \dfrac{1}{2}\right]e^{-t} + t - \dfrac{4}{3} + te^{-t} \\ -\left(\dfrac{c_1}{\sqrt{2}}\right)e^{-3t} + \left[\dfrac{c_2}{\sqrt{2}} - \dfrac{1}{2}\right]e^{-t} + 2t - \dfrac{5}{3} + te^{-t} \end{pmatrix}$$

$$= k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t} + k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

3) **Variation of parameters**

Recall from before, $ay'' + by' + cy = g(t)$

We let $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ here $y_1(t)$ & $y_2(t)$ are fundamental solutions of $ay'' + by' + cy = 0$.

Now for a system： $\dot{x} = p(t)x + g(t)$,

let $\psi(t)$ be a fundamental matrix for $\dot{x} = p(t)x$.

To find a solution for the nonhomogeneous system,

we suppose $x(t) = \psi(t)u(t)$, and we'd like to determine $u(t)$.

Since $x(t)$ satisfies the ODE

$$\therefore \quad \dot{x} = \dot{\psi}(t)u(t) + \psi(t)\dot{u}(t) = p(t)\psi(t)u(t) + g(t)$$

$$\Rightarrow \quad \psi(t)\dot{u}(t) = g(t) \quad \Rightarrow \quad \dot{u}(t) = \psi^{-1}(t)g(t)$$

$$\Rightarrow \quad u(t) = \int_{t_0}^{t} \psi^{-1}(s)g(s)ds + C \qquad C \text{ ： constant vector}$$

$$\Rightarrow \quad x = \psi(t)u(t) = \psi(t)C + \psi(t)\int_{t_0}^{t} \psi^{-1}(s)g(s)ds$$

If we have initial data $x(t_0)=x_0$,

$$\Rightarrow \quad x_0=\psi(t_0)C \quad \Rightarrow \quad C=\psi^{-1}(t_0)x_0$$

$$\therefore \quad x=\psi(t)\psi^{-1}(t_0)x_0+\psi(t)\int_{t_0}^{t}\psi^{-1}(s)g(s)ds$$

- If we use the special fundamental matrix $\Phi(t)$ satisfying $\Phi(t_0)=\mathrm{I}$, then

$$x(t)=\Phi(t)x_0+\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)g(s)ds\,.$$

- If we use $\Phi(t)=e^{\mathrm{A}t}$, $\Phi(0)=\mathrm{I}$

$$\Rightarrow \quad x(t)=e^{\mathrm{A}t}x_0+\int_0^t e^{\mathrm{A}(t-s)}g(s)ds$$

$$\left(\dot{x}=\mathrm{A}x+g(t)\right)$$

Ex： $\dot{x} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathrm{A}x + g(t)$

sol： Since we already solved a fundamental matrix $\psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$.

Then the solution $x(t)$ is given by $x(t) = \psi(t)u(t)$, and we have

$\psi(t)\dot{u}(t) = g($

$$\Rightarrow \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} \dot{u}_1 = e^{2t} - \frac{3}{2} t e^{3t} \\ \dot{u}_2 = 1 + \frac{3}{2} t e^{t} \end{array} \quad \Rightarrow \begin{array}{l} u_1(t) = \frac{1}{2} e^{2t} - \frac{1}{2} t e^{3t} + \frac{1}{6} e^{3t} + c_1 \\ u_2(t) = t + \frac{3}{2} t e^{t} - \frac{3}{2} e^{t} + c_2 \end{array}$$

$\Rightarrow x(t) = \psi(t)u($

$$= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

4) **Laplace transform**

Recall that $L\{x'(t)\} = s\underline{\overline{x}}(s) - x(0)$ here $\underline{\overline{x}}(s) = L\{x(t)\}$

$$\dot{x} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathrm{A}x + g(t)$$

Applying Laplace transform on both sides, we get

$$s\underline{\overline{x}}(s) - x(0) = \mathrm{A}\underline{\overline{x}}(s) + G(s) \quad \text{here} \quad G(s) = L\{g(t)\} = \begin{pmatrix} \dfrac{2}{s+1} \\ \dfrac{3}{s^2} \end{pmatrix}.$$

Suppose the initial data is $x(0) = 0$, then $(s\mathrm{I} - \mathrm{A})\underline{\overline{x}}(s) = G(s)$

$$\Rightarrow \quad \underline{\overline{x}}(s) = (s\mathrm{I} - \mathrm{A})^{-1} G(s)$$

$$\because \quad s\mathrm{I} - \mathrm{A} = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}$$

$$\therefore (s\mathrm{I}-\mathrm{A})^{-1} = \frac{1}{(s+1)(s+3)}\begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix}$$

$$\Rightarrow \quad \underline{\bar{x}}(s) = \begin{pmatrix} \dfrac{2(s+2)}{(s+1)^2(s+3)} + \dfrac{3}{s^2(s+1)(s+3)} \\ \dfrac{2}{(s+1)^2(s+3)} + \dfrac{3(s+2)}{s^2(s+1)(s+3)} \end{pmatrix}$$

$$\Rightarrow \quad x(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Ex：$\begin{cases} y'' + y = g(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

Recall the solution is $y(t) = \int_0^t \sin(t-s)g(s)ds$

Let us try to use e^{At} to find the solution formula.

sol：

step1： Write this equation as a 2×2 system.

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}_2 &= \ddot{y} = -y + g(t) = -x_1 + g(t) \\ \dot{x}_1 &= x_2 \end{aligned}$$

$$\therefore \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

$$\dot{x} = Ax + h \qquad \therefore \; x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}h(s)ds$$

step2： What is $e^{\mathrm{A}t}$?

Here we will not use the Jordon form to compute, instead, we solve the system

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathrm{A}x \\ x_1(0) = x_{10} \\ x_2(0) = x_{20} \end{cases}$$

then the solution can be written as $e^{\mathrm{A}t}\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$.

On the other hand, we can solve the system easily.

$$\ddot{x}_1 + x_1 = 0 \qquad \therefore\ x_1(t) = c_1 \sin t + c_2 \cos t$$

$$\Rightarrow \quad x_2(t) = \dot{x}_1(t) = c_1 \cos t - c_2 \sin t$$

$$x_1(0) = c_2 = x_{10} \qquad\qquad x_1(t) = x_{20} \sin t + x_{10} \cos t$$

$$x_2(0) = c_1 = x_{20} \qquad \therefore \quad x_2(t) = x_{20} \cos t - x_{10} \sin t$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = e^{\mathrm{A}t} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

$$\therefore \quad e^{\mathrm{A}t} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$\therefore$ For the nonhomogeneous problem, we have

$$x(t) = e^{\mathrm{A}t} x_0 + \int_0^t e^{\mathrm{A}(t-s)} h(s) ds$$

$$= \int_0^t \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix} \begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} \sin(t-s) g(s) \\ \cos(t-s) g(s) \end{pmatrix} ds = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Since we are interested in the first component,

then $x_1(t) = y(t) = \int_0^t \sin(t-s) g(s) ds$.