9.3 Locally linear system

In the previous sections, we have described the stability properties of the equilibrium solution $x = \mathbf{0}$ of the 2D linear system

$$\dot{x} = \mathrm{A}x \qquad (\mathrm{A} \ \text{nonsingular})$$

Theorem 9.3.1

The critical point $x = 0$ of the linear system $\dot{x} = Ax$ is

(1) asymptotically stable (A.S.) if the eigenvalues r_1 , r_2 are real and negative or have negative real parts;

(2) stable but not A.S. if r_1 & r_2 are pure imaginary;

(3) unstable if r_1 & r_2 are real and either is positive, or if they have positive real part.

Q： How about the stability issue for nonlinear systems ?

$$\begin{cases} \dot{x} = f(x) \\ f(0) = 0 \end{cases}$$

$x = 0$ is an isolated critical point.

We consider the case

$$\begin{cases} \dot{x} = \mathrm{A}x + g(x) \\ x = 0 \end{cases}$$

$x = 0$ is a critical point.

Assume the components of g have continuous first partial derivatives and $\dfrac{\|g(x)\|}{\|x\|} \to 0$ as $x \to 0$,

then such a system is called an **almost linear system** in the neighborhood of the critical point $x = 0$.

Ex：

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 \sin x - ry = -\omega^2 x + \dfrac{1}{3!}\omega^2 x^3 + \cdots \end{cases}$$

(motion of a pendulum)

Show that the system is locally linear near the origin.

sol：

The critical points are (x, y) such that

$$\begin{cases} y = 0 \\ -\omega^2 \sin x - ry = 0 = -\omega^2 \sin x \end{cases}$$

$\therefore\ x = \pm n\pi \ \rightarrow$ isolated critical point

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\underbrace{\begin{pmatrix}0 & 1\\ -\omega^2 & -r\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}}_{linear}-\omega^2\underbrace{\begin{pmatrix}0\\ \sin x - x\end{pmatrix}}_{nonlinear}$$

$$g_1(x,y)=0$$

$$g_2(x,y)=-\omega^2(\sin x - x)\approx -\omega^2\frac{x^3}{3!} \quad \text{as} \quad x\to 0$$

$$(x=r\cos\theta \qquad y=r\sin\theta)$$

$$\therefore \quad \frac{\|g_2(x,y)\|}{\|(x,y)\|}\cong \frac{-\omega^2 r^3\cos^3\theta}{3!}\Big/ r \quad \to \quad 0 \quad \text{as} \quad r\to 0$$

$\therefore$ It is an almost linear system near the origin.

9.3 #8

$$\begin{cases} \dot{x} = x - x^2 - xy \\ \dot{y} = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy \end{cases} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^2 - xy \\ \frac{1}{4}y^2 - \frac{3}{4}xy \end{pmatrix}$$

$$g_1(x,y) = x^2 - xy = r^2(\cos^2\theta - \sin\theta\cos\theta)$$

$$g_2(x,y) = \frac{1}{4}\left(r^2\sin^2\theta\right) - \frac{3}{4}r^2\sin\theta\cos\theta = \frac{1}{4}r^2\sin\theta\left(\sin\theta - 3\cos\theta\right)$$

$$\therefore \quad \frac{\|g(x)\|}{\|x\|} = \frac{\sqrt{g_1{}^2 + g_2{}^2}}{r} \to 0 \quad \text{as } r \to 0$$

$$\begin{cases} x - x^2 - xy = x(1 - x - y) \\ \frac{1}{4}y\left(2 - y - 3x\right) = 0 \end{cases} \quad \begin{aligned} &\Rightarrow \quad x = 0 \quad or \quad x + y = 1 \\ &\Rightarrow \quad y = 0 \quad or \quad 3x + y = 2 \end{aligned}$$

① $x = 0 \quad , \quad y = 0$

② $x = 0 \quad , \quad 3x + y = 2 \quad \Rightarrow \quad y = 2 \qquad \therefore \quad (0, 2)$

③ $y = 0 \quad , \quad x + y = 1 \quad \Rightarrow \quad (x, y) = (1, 0)$

④ $x + y = 1 \quad , \quad 3x + y = 2 \quad \Rightarrow \quad 2x = 1 \quad \Rightarrow \quad x = \frac{1}{2} \quad , \quad y = \frac{1}{2}$

Theorem 9.3.2

$$\dot{x} = \mathrm{A}x \quad (1) \qquad\qquad \dot{x} = \mathrm{A}x + g(x) \quad (2)$$

Let r_1 & r_2 be the eigenvalues of the linear system (1) corresponding to the locally linear system (2).

If both r_1 & r_2 have nonzero real parts, then the stabilities of the fixed point $x = 0$ are the same in (1) & (2).

- Moral of the story：

To determine the stability of $x = 0$ for (2), we just need to determine the stability of $x = 0$ for (1) if the eigenvalues have nonzero real parts.

Nonlinear system and stability — general setting

Given a nonlinear system： $\dot{x} = f(x)$, $x \in \mathbb{R}^n$

Let x_0 be a fixed point, i.e. $f(x_0) = 0$.

Q： How do we study the stability of the fixed point？

We want to understand how solutions behave near a special solution.

(1) Solve the system if we can.

(2) Polar coordinates for $n = 2$.

(3) Linearization.

(4) Liapunov function.

(5) Numerical simulation.

Recall

Taylor's theorem in two variables

$$f(x+h, y+k)$$

$$= f(x,y) + hf_x(x,y) + kf_y(x,y) + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}) + \cdots$$

Ex： Find the linearization of $\begin{cases} \dot{x} = x - y^2 \\ \dot{y} = x - 2y + 2x^2 \end{cases}$ around the fixed point $(0,0)$

and determine the stability of the critical point by the linearized system.

sol：

solve $\begin{cases} x - y^2 = 0 & \Rightarrow \quad x = y^2 \\ x - 2y + 2x^2 = 0 & \Rightarrow \quad y^2 - 2y + 2y^4 = 0 \end{cases}$

$\Rightarrow \quad 2y^4 + y^2 - 2y = y(2y^3 + y - 2) = 0$

$(0,0)$ is a fixed point.

$$\dot{x} = f(x,y) = x - y^2 \quad , \quad \dot{y} = g(x,y) = x - 2y + 2x^2$$

The linearization around $(0,0)$ is

$$\dot{x} = f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y \qquad \therefore \quad \frac{\partial f}{\partial x} = 1 \quad , \quad \frac{\partial f}{\partial y} = -2y$$

$$\dot{y} = g(x,y) = g(0,0) + \frac{\partial g}{\partial x}(0,0)x + \frac{\partial g}{\partial y}(0,0)y \qquad \Rightarrow \quad \frac{\partial g}{\partial x} = 1 + 4x \quad , \quad \frac{\partial g}{\partial y} = -$$

$$\therefore \quad \begin{cases} \dot{x} = x \\ \dot{y} = x - 2y \end{cases}$$ is the linearized system.

Check the eigenvalues of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \det \begin{pmatrix} 1-\lambda & 0 \\ 1 & -2-\lambda \end{pmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 + \lambda - 2 = 0 \quad \Rightarrow \quad (\lambda+2)(\lambda-1) = 0$$

$$\Rightarrow \quad \lambda = -2, \ \ 1$$

$\therefore$ The linearized system has an unstable saddle point at $(0,0)$,

so does the nonlinear system.

Nonlinear system and stability

We begin with linearization at a fixed point.

Ex： Show that the system $\begin{cases} \dot{x}_1 = e^{x_1+x_2} - x_2 = f(x_1, x_2) \\ \dot{x}_2 = -x_1 + x_1 x_2 = g(x_1, x_2) \end{cases}$ has only one fixed point.

Find the linearization at that point.

sol：

To find the fixed points, we solve $\begin{cases} e^{x_1+x_2} - x_2 = 0 & (1) \\ -x_1 + x_1 x_2 = 0 & (2) \end{cases}$

Form (2), we get $-x_1 + x_1 x_2 = 0 = x_1(x_2 - 1)$

$\therefore \quad x_1 = 0 \quad or \quad x_2 = 1$

- $x_1 = 0$, substitute into (1), we get $e^{x_2} - x_2 = 0 \quad \Rightarrow \quad$ no solution
- $x_2 = 1, \quad e^{x_1+1} - 1 = 0 \qquad \therefore \quad x_1 = -1$

$\therefore$ The only fixed point is $(-1,1)$.

In order to study stability of $(-1,1)$, we try to find the linearization at that point.

$$\dot{x}_1 = e^{x_1+x_2} - x_2 = f(x_1, x_2) = f(-1,1) + \frac{\partial f}{\partial x_1}(-1,1)\left(x_1+1\right) + \frac{\partial f}{\partial x_2}(-1,1)\left(x_2-1\right) + 0\left(r^2\right)$$

$$= \left(x_1 - 1\right) + 0\left(r^2\right)$$

$$\dot{x}_2 = -x_1 + x_1 x_2 = g(-1,1) + \frac{\partial g}{\partial x_1}(-1,1)\left(x_1+1\right) + \frac{\partial g}{\partial x_2}(-1,1)\left(x_2-1\right) + 0\left(r^2\right)$$

$$= -\left(x_2 - 1\right) + 0\left(r^2\right)$$

Since we want to know the stability $(-1,1)$, i.e.,

we'd like to know what happens to the point near $(-1,1)$.

So it is convenient to introduce the change of variables

$$y_1 = x_1 + 1 \qquad y_2 = x_2 - 1$$

Then the equations of y_1 and y_2 become

$$\dot{y}_1 = y_1 \qquad \dot{y}_2 = -y_2 \qquad \text{(linearized)}$$

$\therefore \ (x_1, x_2) = (-1,1) \iff (y_1, y_2) = (0,0)$ is a saddle point.

(unstable)

$$\dot{x}_1 = f(x_1, x_2) \qquad \dot{x}_2 = g(x_1, x_2)$$

If $f(x_1, x_2) = 0 = g(x_1, x_2)$, then the linearization of this system around the fixed point is

$$\dot{x} = Ax = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \end{bmatrix}_{(0,0)} x$$

Q： What happens if the fixed point is at (c_1, c_2) ?

$$f(c_1,c_2)=0=g(c_1,c_2)$$

Then the linearization around (c_1,c_2) is $\dot{x}=\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix}_{(c_1,c_2)} \begin{pmatrix} x_1-c_1 \\ x_2-c_2 \end{pmatrix}$

If we introduced the variables $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} x_1-c_1 \\ x_2-c_2 \end{pmatrix} \Rightarrow \dot{y}=\mathrm{A}y=\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix}_{(c_1,c_2)} y$

HW：

Find the linearization of the following systems at the indicated fixed points.

i) $$\begin{cases} \dot{x}_1 = x_1 + \dfrac{x_1 x_2{}^3}{(1+x_1{}^2)^2} \\ \dot{x}_2 = 2x_1 - 3x_2 \end{cases} , \quad at \quad (0,0)$$

ii) $$\begin{cases} \dot{x}_1 = x_1{}^2 + \sin x_2 - 1 \\ \dot{x}_2 = \sinh(x_1 - 1) \end{cases} , \quad at \quad (1,0)$$

Q： After we know how to linearize equations, what can we do with the linearized system？

Linearization Theorem

Let the nonlinear system $\dot{x} = F(x)$ has a simple fixed point at $x = 0$.

Then in a neighborhood of the origin the phase portraits of the system and its linearization are qualitatively equivalent provided the linearized system is not a center. (Hartman-Grobman Theorem)

Q： What is the moral of the story？

Remark：

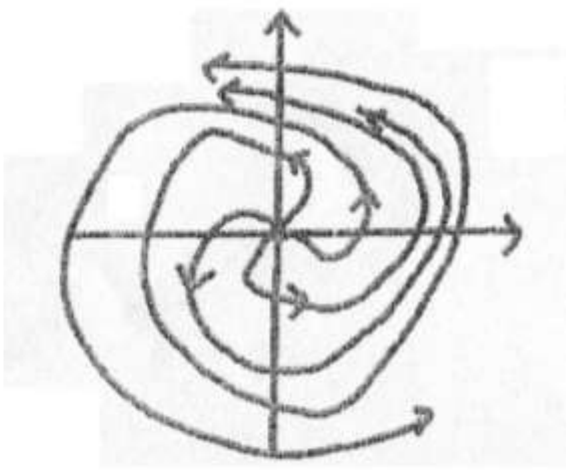
If the eigenvalues of the linearized system have non-zero real part, the phase portraits of the nonlinear system and its linearization are qualitatively equivalent in a neighborhood of the fixed point.

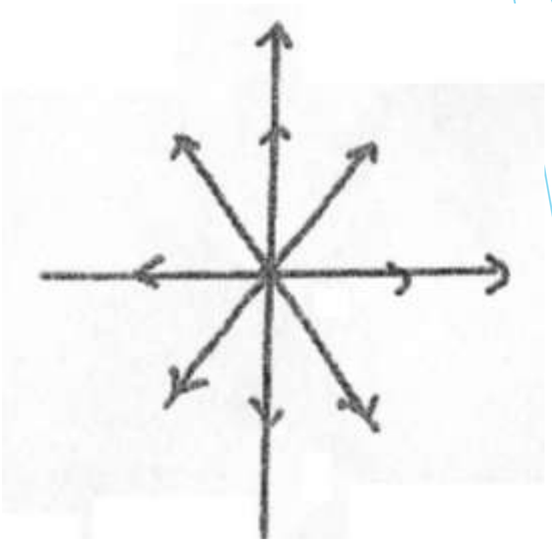
Such fixed points are said to be **hyperbolic.**

Hyperbolic fixed points are robust！

Centers are easy to change！

Ex： Nonlinear (Hyperbolic) Linearized





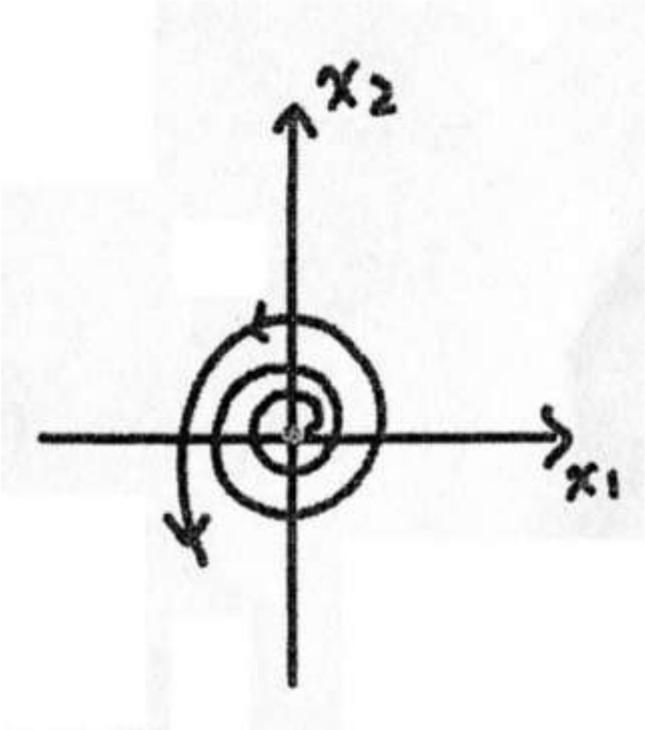
$$\{\dot{x}_1 = x_1 - x_2^{\,3},\quad \dot{x}_2 = x_2 + x_1^{\,3}\}$$

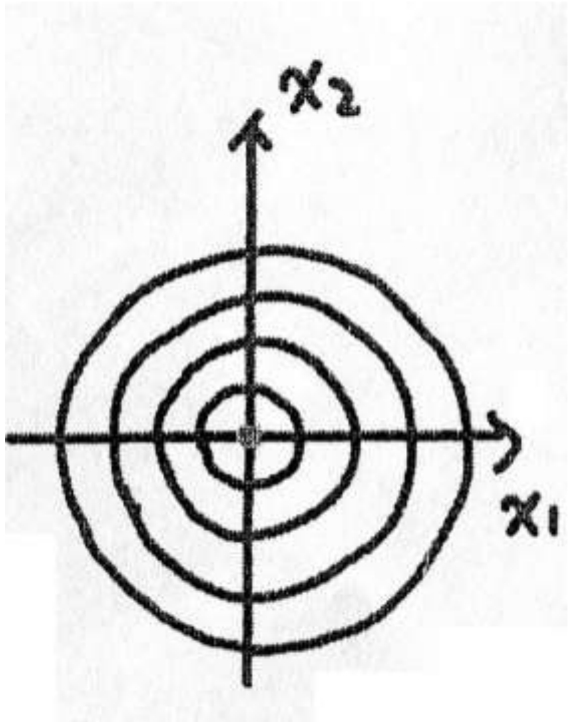
$$\{\dot{x}_1 = x_1,\quad \dot{x}_2 = x_2\}$$

For the nonlinear system, the stability of a hyperbolic fixed point can be determined by the linearized system.

Q： How about a center？

Ex：



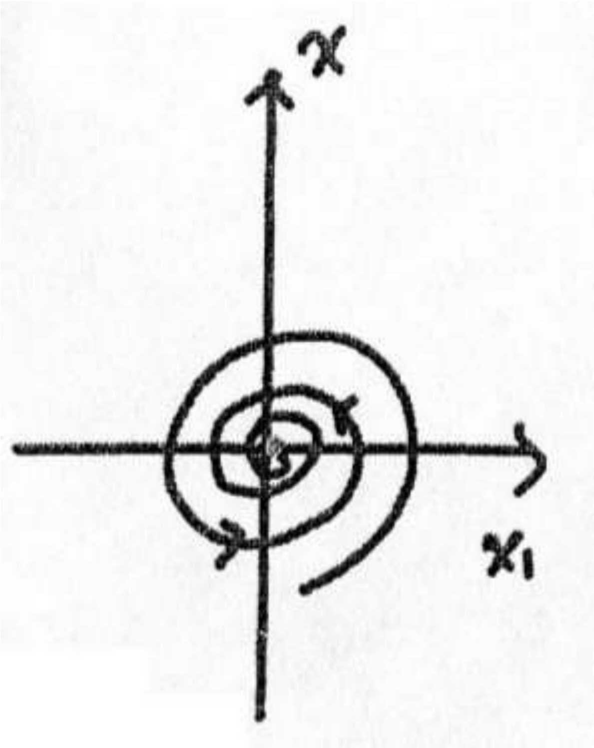


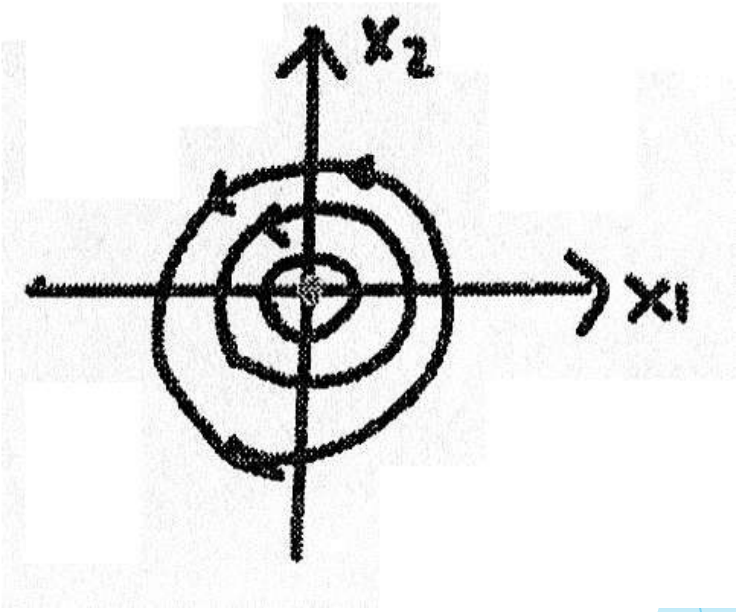
$$\dot{x}_1 = -x_2 + x_1(x_1^{\,2} + x_2^{\,2})$$
$$\dot{x}_2 = x_1 + x_2(x_1^{\,2} + x_2^{\,2})$$

(Nonlinear)

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1$$

(Linearization)





$$\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 - x_2(x_1^2 + x_2^2)$$

(Nonlinear)

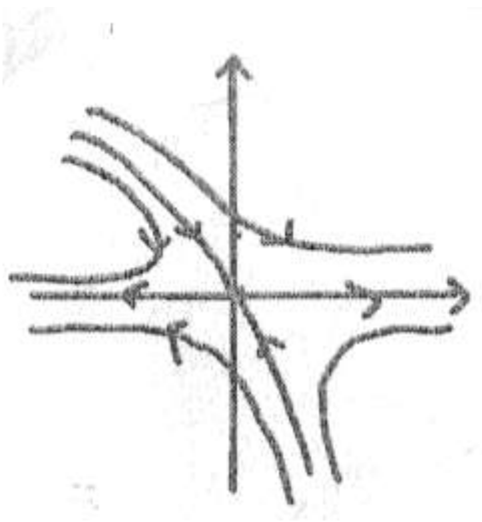
$$\dot{x}_1 = -x_2 - x_2(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -x_1 + x_1(x_1^2 + x_2^2)$$

(Nonlinear)

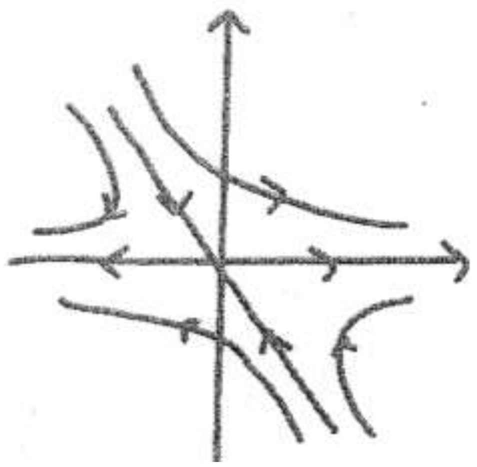
Ex： Use the linearization theorem to determine the local phase portrait

of the system $\begin{cases} \dot{x}_1 = x_1 + 4x_2 + e^{x_1} - 1 \\ \dot{x}_2 = -x_2 - x_2 e^{x_1} \end{cases}$ at the origin.

sol：



nonlinear



Note $(0,0)$ is a fixed point.

The linearization around $(0,0)$ is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \end{pmatrix}_{(0,0)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1+e^{x_1} & 4 \\ -x_2 e^{x_1} & -1-e^{x_1} \end{pmatrix}_{(0,0)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\lambda = 2,\ -2 \quad \Rightarrow$ saddle point

$\therefore\ (0,0)$ is also an unstable fixed point for the nonlinear system.

Ex： Use the linearization theorem to classify the fixed points of the system

$$\begin{cases} \dot{x}_1 = \sin(x_1 + x_2) = f \\ \dot{x}_2 = x_2 = g \end{cases}$$

sol：

First solve the fixed point of the system $\begin{cases} \sin(x_1 + x_2) = 0 \\ x_2 = 0 \end{cases}$

$$\sin x_1 = 0 \quad \Rightarrow \quad x_1 = \pm n\pi \qquad \therefore (x_1, x_2) = (\pm n\pi, 0)$$

$$A = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \cos(x_1 + x_2) & \cos(x_1 + x_2) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \ (-1-\lambda)(1-\lambda) = 0 \quad \Rightarrow \quad \lambda = -1,\ 1 \quad \Rightarrow \quad \text{unstable}$$

HW：

Use the linearization theorem to classify, where possible, the fixed points of the systems：

a)
$$\begin{cases} \dot{x}_1 = x_1 - x_2 - e^{x_1} \\ \dot{x}_2 = x_1 - x_2 - 1 \end{cases}$$

b)
$$\begin{cases} \dot{x}_1 = -x_2 + x_1 + x_1 \\ \dot{x}_2 = x_1 - x_2 - x_2^2 \end{cases}$$