Chapter 7.

Systems of first order linear equations.

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- The mathematical problems for many physical phenomenon consist of a system of two or more differential equations.

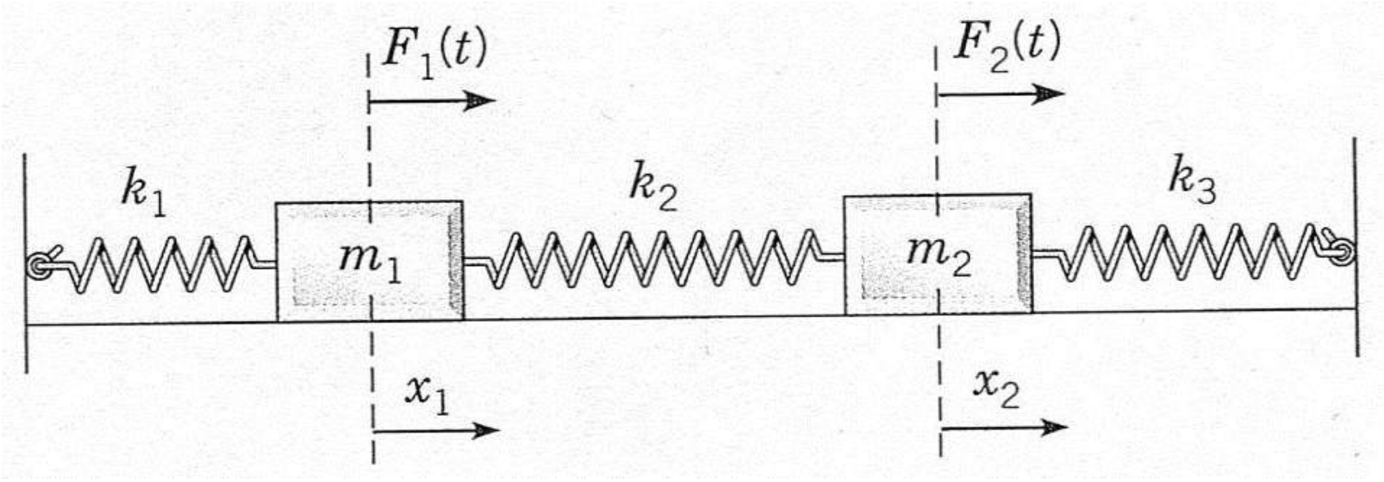
- In this chapter, we show you how to solve this kind of problems.

7.1 Introduction

- In this section, we first give some simple examples then give a brief introduction on the systems of differential equations.

Example 1

Consider the spring-mass system.



- Tow masses move on a frictionless surface under the influence of external forces $F_1(t)$ and $F_2(t)$, and they are also constrained by the three springs whose constants are k_1, k_2 and k_3.

- Then the system of differential equations describing this physical system can be written as

$$\begin{cases} m_1 \dfrac{d^2 x_1}{dt^2} = k_2(x_2 - x_1) - \mathrm{k}_1 x_1 + F_1(\mathrm{t}) \\ \qquad\qquad = -(\mathrm{k}_1 + \mathrm{k}_2) x_1 + k_2 x_2 + F_1(\mathrm{t}) \\ m_2 \dfrac{d^2 x_2}{dt^2} = -\mathrm{k}_3 x_2 - k_2(x_2 - x_1) + F_2(\mathrm{t}) \\ \qquad\qquad = k_2 x_1 - (\mathrm{k}_2 + \mathrm{k}_3) x_2 + F_2(\mathrm{t}) \end{cases}$$

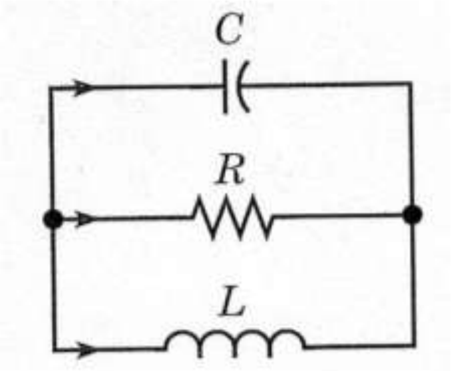
Where t : independent variables

x_1, x_2 : dependent variables.

$x_1(\mathrm{t}), x_2(\mathrm{t})$

Example 2

- Consider the parallel LRC circuit, let V be the voltage drop across the capacitor and I be the current through the inductor.



- Then the system of equations can be written by

$$\begin{cases} \dfrac{dI}{dt} = \dfrac{V}{L} \\ \dfrac{dV}{dt} = -\dfrac{I}{C} - \dfrac{V}{RC} \end{cases}$$

where L is the inductor,
C is the capacitance,
R is the resistance.

Basic theory

Def： The system of first order equation is of the form：

$$\begin{aligned}
\dot{x}_1 &= F_1(\mathrm{t}, x_1, x_2, \ldots, x_n) \\
\dot{x}_2 &= F_2(\mathrm{t}, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\dot{x}_n &= F_n(\mathrm{t}, x_1, x_2, \ldots, x_n)
\end{aligned}$$

- If each of the functions $F_1, \ldots, F_n$ is a linear function of the dependent variables $x_1, x_2, \ldots, x_n$, then the system of equations is said to be linear；Otherwise, it is nonlinear.

∴ The most general system of n first order linear equations has the form

$$\begin{aligned}
\dot{x}_1 &= p_{11}(\mathrm{t})x_1 + \ldots + p_{1n}(\mathrm{t})x_n + g_1(\mathrm{t}) \\
\dot{x}_2 &= p_{21}(\mathrm{t})x_1 + \ldots + p_{2n}(\mathrm{t})x_n + g_2(\mathrm{t}) \\
&\vdots \\
\dot{x}_n &= p_{n1}(\mathrm{t})x_1 + \ldots + p_{nn}(\mathrm{t})x_n + g_n(\mathrm{t})
\end{aligned}$$

- If each of the functions $g_1(\mathrm{t}), \ldots, g_n(\mathrm{t})$ is zero $\forall t \in I$, then the system is **homogeneous**；Otherwise, it is **non-homogeneous**.

Remark：

One reason why system of first order equations are particularly important is that equations of higher order can always be transformed into such systems.

Ex： $u'' + 0.125u' + u = 0$, Rewrite this equation as a system of first order equations.

sol： Let $x_1 = u$ and $x_2 = \dot{u}$. We try to derivate a system of two equations for x_1 & x_2 from the ODE.

$$\because \begin{aligned} \dot{x}_1 &= \dot{u} = x_2 \\ \dot{x}_2 &= \ddot{u} = -0.125\dot{u} - u = -0.125x_2 - x_1 \end{aligned}$$

$$\therefore \; u'' + 0.125u' + u = 0 \quad \Leftrightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - 0.125x_2 \end{cases}$$

(second order ODE) (system of two first order ODEs).

Generalization

$$
\begin{array}{cccc}
x_1 & x_2 & & x_n \\
\downarrow & \downarrow & \cdots & \downarrow
\end{array}
$$

$$y^{(n)} = F(t, y, y', \ldots y^{(n-1)}) \quad \Leftrightarrow \quad \text{system of n first order ODEs.}$$

(nth order ODE) How?

- To transform an arbitrary nth order equation into a system of n first order equations, we introduce the variable $x_1, x_2, \ldots, x_n$ defined by

$$
\begin{aligned}
x_1 &= y \\
x_2 &= \dot{y} \\
&\vdots \\
x_n &= y^{(n-1)}
\end{aligned}
$$

- To find the system of differential equations for $x_1, x_2, \ldots, x_n$, we just differentiate and get

$$\begin{aligned}
\dot{x}_1 &= \dot{y} = x_2 \\
\dot{x}_2 &= \ddot{y} = x_3 \\
&\vdots \\
\dot{x}_{n-1} &= (y^{(\mathrm{n}-2)})' = y^{(\mathrm{n}-1)} = x_n \\
\dot{x}_n &= y^{(\mathrm{n})} = F(\mathrm{t}, \mathrm{y}, \dot{\mathrm{y}}, \ldots \mathrm{y}^{(\mathrm{n}-1)}) \\
&\qquad\quad = F(\mathrm{t}, x_1, x_2, \ldots, x_n)
\end{aligned}$$

$\therefore$ The transformed system is

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= F(\mathrm{t}, x_1, x_2, \ldots, x_n)
\end{aligned}$$

Basic theory：

Solutions of First Order Systems

- A system of simultaneous first order ordinary differential equations has the general form

$$
\begin{aligned}
x_1' &= F_1(\mathrm{t}, x_1, x_2, \ldots, x_n) \\
&\vdots \\
x_n' &= F_n(\mathrm{t}, x_1, x_2, \ldots, x_n)
\end{aligned}
$$

It has a **solution** on I： $\alpha < t < \beta$ if there exists n functions

$$x_1 = \phi_1(\mathrm{t}),\ x_2 = \phi_2(\mathrm{t}),\ \ldots,\ x_n = \phi_n$$

that are differentiable on I and satisfy the system of equations at all points t in I.

- Initial conditions may also be prescribed to give an IVP：

$$x_1(\mathrm{t}_0) = x_1^0,\ x_2(\mathrm{t}_0) = x_2^0,\ \ldots,\ x_n(\mathrm{t}_0) = x_n^0$$

Theorem 7.1.1

- Suppose $F_1,\dots,F_n$ and $\frac{\partial F_1}{\partial x_1},\dots,\frac{\partial F_1}{\partial x_n},\dots,\frac{\partial F_n}{\partial x_1},\dots,\frac{\partial F_n}{\partial x_n}$, are continuous in the region R of $t\,x_1x_2\dots x_n$-space defined by

$$\alpha < t < \beta, \alpha_1 < x_1 < \beta_1, \dots, \alpha_n < x_n < \beta_n,$$

and let the point $(\mathrm{t}_0, x_1^0, x_2^0, \dots, x_n^0)$

be contained in R. Then in some interval $(\mathrm{t}_0 - \mathrm{h}, \mathrm{t}_0 + \mathrm{h})$ there exists a unique solution

$$x_1 = \phi_1(\mathrm{t}), x_2 = \phi_2(\mathrm{t}), \dots, x_n = \phi_n$$

that satisfies the IVP.

$$\begin{aligned} x_1' &= F_1(\mathrm{t}, x_1, x_2, \dots, x_n) \\ x_2' &= F_2(\mathrm{t}, x_1, x_2, \dots, x_n) \\ &\vdots \\ x_n' &= F_n(\mathrm{t}, x_1, x_2, \dots, x_n) \end{aligned}$$

Linear Systems

- If each F_k is a linear function of $x_1, x_2, \ldots, x_n$, then the system of equations has the general form

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t)
\end{aligned}
$$

- If each of the $g_k(t)$ is zero on I, then the system is **homogeneous**, otherwise, it is **non-homogeneous.**

Theorem 7.1.2

- Suppose $p_{11}, p_{12}, \ldots, p_{nn}, g_1, \ldots, g_n$ are continuous on an interval I： $\alpha < t < \beta$ with t_0 in I, and let

$$x_1^0, x_2^0, \ldots, x_n$$

prescribe the initial conditions. Then there exists a unique solution

$$x_1 = \phi_1(t), x_2 = \phi_2(t), \ldots, x_n = \phi_n$$

that satisfies the IVP, and exists throughout I.

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}x_n + g_1(t) \\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}x_n + g_2(t) \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}x_n + g_n(t)
\end{aligned}
$$

7.2 - 7.3 Review of matrices.

We will skip most topics on basic theory in linear algebra and will talk about only some important ones.

- Let A be an $m \times n$ matrix.

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- transpose of A： A^T

$$\mathrm{A} = \left(a_{ij}\right) \quad \Rightarrow \quad \mathrm{A}^T = \left(a_{ji}\right)$$

- Conjugate of A： $\overline{\mathrm{A}}$

$$\overline{\mathrm{A}} = \left(\overline{a}_{ij}\right)$$

- adjoint of A： $\overline{\mathrm{A}}^T = \left(\overline{a}_{ji}\right) = \mathrm{A}^*$

Boyce/DiPrima 10th ed, Ch 7.3: Systems of Linear Equations, Linear Independence, Eigenvalues

- A system of *n* linear equations in *n* variables,

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\
&\vdots \\
a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n ,
\end{aligned}
$$

  can be expressed as a matrix equation **Ax** = **b**:

$$
\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}
$$

- If **b** = **0**, then system is **homogeneous**; otherwise it is **nonhomogeneous**.

Nonsingular Case

- If the coefficient matrix **A** is nonsingular, then it is invertible and we can solve **Ax** = **b** as follows:

$$\mathbf{Ax} = \mathbf{b} \;\Rightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \;\Rightarrow \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b} \;\Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

- This solution is therefore unique. Also, if **b** = **0**, it follows that the unique solution to **Ax** = **0** is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$.

- Thus if **A** is nonsingular, then the only solution to **Ax** = **0** is the trivial solution **x** = **0**.

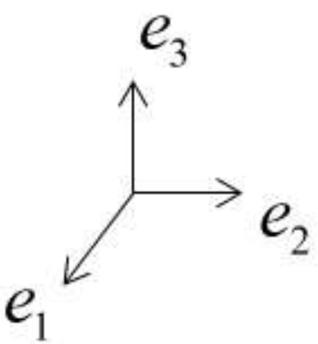
Singular Case

- If the coefficient matrix $\mathbf{A}$ is singular, then $\mathbf{A}^{-1}$ does not exist, and either a solution to $\mathbf{Ax} = \mathbf{b}$ does not exist, or there is more than one solution (not unique).
- Further, the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has more than one solution. That is, in addition to the trivial solution $\mathbf{x} = \mathbf{0}$, there are infinitely many nontrivial solutions.
- The nonhomogeneous case $\mathbf{Ax} = \mathbf{b}$ has no solution unless $(\mathbf{b}, \mathbf{y}) = 0$, for all vectors $\mathbf{y}$ satisfying $\mathbf{A}^*\mathbf{y} = \mathbf{0}$, where $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$.
- In this case, $\mathbf{Ax} = \mathbf{b}$ has solutions (infinitely many), each of the form $\mathbf{x} = \mathbf{x}^{(0)} + \boldsymbol{\xi}$, where $\mathbf{x}^{(0)}$ is a particular solution of $\mathbf{Ax} = \mathbf{b}$, and $\boldsymbol{\xi}$ is any solution of $\mathbf{Ax} = \mathbf{0}$.

Eigenvalues and eigenvectors

- Diagonal matrix $A=\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$

$$e_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad e_2=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad e_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$$



$$\Rightarrow\quad Ae_1=a_{11}e_1,\; Ae_2=a_{22}e_2,\; Ae_3=a_{33}e_3$$

Q： What about an arbitrary matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad ?$$

then clearly $Ae_j \neq a_{11} e_j$.

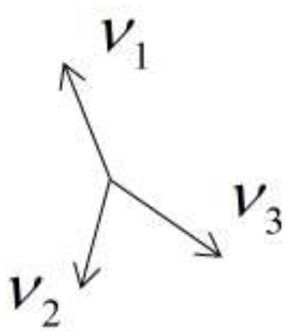
Def： λ is an eigenvalue of A if $\exists$ nonzero solution v

$\ni$ $Av = \lambda v \quad (A: n \times n, \ v \in \mathbf{R}^n)$,

where λ can be complex numbers, and v is called the eigenvector corresponding to that eigenvalue.

Remark：

- Let A be a 3×3 matrix.
- Suppose $\exists$ $v_1, v_2, v_3, \lambda_1, \lambda_2, \lambda_3$ $\ni$ $Av_1=\lambda_1 v_1$, $Av_2=\lambda_2 v_2$, $Av_3=\lambda_3 v_3$.
- Then A "looks like" a diagonal matrix in the coordinates $\{v_1, v_2, v_3\}$.



Q： How to find λ & v $\ni$ $Av = \lambda v$?

$\Rightarrow$ $Av - \lambda I v = 0$ $\Rightarrow$ $(A - \lambda I)v = 0$

$\therefore$ If $\exists$ a nonzero solution v

$\Rightarrow$ $A - \lambda I$ is singular $\Rightarrow$ $\det(A - \lambda\ I) =$

After λ is found, we can find v by solving $Av = \lambda v$.

Def：

The equation $\det(A - \lambda I) = 0$ is a polynomial of degree n in λ and is called the characteristic equation of the matrix A.

Example 4: Eigenvalues (1 of 3)

- Find the eigenvalues and eigenvectors of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

- Solution: Choose λ such that det(**A**-λ**I**) = 0, as follows.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-2-\lambda) - (-1)(4)$$

$$= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

$$\Rightarrow \lambda = 2, \lambda = -1$$

Example 4: First Eigenvector (2 of 3)

- To find the eigenvectors of the matrix **A**, we need to solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda = 2$ and $\lambda = -1$.
- Eigenvector for $\lambda = 2$: Solve

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0} \Leftrightarrow \begin{pmatrix} 3-2 & -1 \\ 4 & -2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and this implies that $x_1 = x_2$. So

$$\mathbf{x}^{(1)} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ c \text{ arbitrary} \rightarrow \text{choose}\, \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Example 4: Second Eigenvector (3 of 3)

- Eigenvector for λ = -1: Solve

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0} \Leftrightarrow \begin{pmatrix} 3+1 & -1 \\ 4 & -2+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and this implies that $x_2 = 4x_1$. So

$$\mathbf{x}^{(2)} = \begin{pmatrix} x_1 \\ 4x_1 \end{pmatrix} = c\begin{pmatrix} 1 \\ 4 \end{pmatrix},\ c \text{ arbitrary} \rightarrow \text{choose } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Normalized Eigenvectors

- From the previous example, we see that eigenvectors are determined up to a nonzero multiplicative constant.

- If this constant is specified in some particular way, then the eigenvector is said to be **normalized**.

- For example, eigenvectors are sometimes normalized by choosing the constant so that $||\mathbf{x}|| = (\mathbf{x}, \mathbf{x})^{1/2} = 1$.

Algebraic and Geometric Multiplicity

- In finding the eigenvalues λ of an n x n matrix **A**, we solve det(**A**-λ**I**) = 0.
- Since this involves finding the determinant of an n x n matrix, the problem reduces to finding roots of an nth degree polynomial.
- Denote these roots, or eigenvalues, by $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- If an eigenvalue is repeated m times, then its **algebraic multiplicity** is m.
- Each eigenvalue has at least one eigenvector, and a eigenvalue of algebraic multiplicity m may have q linearly independent eigevectors, $1 \leq q \leq m$, and q is called the **geometric multiplicity** of the eigenvalue.

Eigenvectors and Linear Independence

- If an eigenvalue λ has algebraic multiplicity 1, then it is said to be **simple**, and the geometric multiplicity is 1 also.
- If each eigenvalue of an n x n matrix **A** is simple, then **A** has n distinct eigenvalues. It can be shown that the n eigenvectors corresponding to these eigenvalues are linearly independent.
- If an eigenvalue has one or more repeated eigenvalues, then there may be fewer than n linearly independent eigenvectors since for each repeated eigenvalue, we may have $q < m$. This may lead to complications in solving systems of differential equations.

Hermitian Matrices

- A **self-adjoint**, or **Hermitian** matrix, satisfies $\mathbf{A} = \mathbf{A}^*$, where we recall that $\mathbf{A}^* = \bar{\mathbf{A}}^T$.
- Thus for a Hermitian matrix, $a_{ij} = \bar{a}_{ji}$.
- Note that if **A** has real entries and is symmetric (see last example), then **A** is Hermitian.
- An n x n Hermitian matrix **A** has the following properties:
  - All eigenvalues of **A** are real.
  - There exists a full set of n linearly independent eigenvectors of **A**.
  - If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are eigenvectors that correspond to different eigenvalues of **A**, then $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are orthogonal.
  - Corresponding to an eigenvalue of algebraic multiplicity m, it is possible to choose m mutually orthogonal eigenvectors, and hence A has a full set of n linearly independent orthogonal eigenvectors.