7.6 Complex eigenvalues

In this section, we consider the linear system $\dot{x} = \mathrm{A}x$
where A is a $n \times n$ real matrix and has complex eigenvalues.

To find the eigenvalues, we solve the following equation

$$\det(\mathrm{A} - \lambda \mathrm{I}) = 0$$

If λ is a complex number, then we also have

$$\det(\mathrm{A} - \bar{\lambda}\,\mathrm{I}) =$$

since everything else is real. $\Rightarrow$ $\bar{\lambda}$ is also an eigenvalue.

To find the eigenvectors, we solve $\mathrm{A}v = \lambda v$

$$\Rightarrow \quad \mathrm{A}\bar{v} = \bar{\lambda}\bar{v}$$

$\therefore$ If v is an eigenvector associated with λ,

then $\bar{v}$ is an eigenvector associated with $\bar{\lambda}$.

Q： What happens if A is not real ?

To solve the ODE, we use an example to illustrate the idea.

Ex： Find a fundamental set of real-valued solutions of the system

$$\dot{x} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} x$$

sol： Let's try to find the eigenvalues first solve $(-\frac{1}{2} - \lambda)^2 + 1 = 0$

$$\therefore \quad \lambda = -\frac{1}{2} \pm i$$

To solve the corresponding eigenvectors,

$$\mathrm{A}v = \left(-\frac{1}{2} + i\right)v \quad \Rightarrow \quad \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \left(-\frac{1}{2} + i\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Rightarrow \quad -\frac{1}{2}v_1 + v_2 = -\frac{1}{2}v_1 + iv_1 \qquad \therefore \quad v_2 = iv_1$$

$\therefore$ The first eigenvector corresponding to the eigenvalue is $\begin{pmatrix} 1 \\ i \end{pmatrix}$.

$\therefore$ To find the second, we have $\begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \left(-\frac{1}{2} - i \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\therefore \quad -\frac{1}{2} v_1 + v_2 = -\frac{1}{2} v_1 - i v_1 \qquad \therefore \quad v_2 = -i v_1$

The second eigenvector is $\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

$\therefore$ The two independent solutions are the following

$$e^{\left(-\frac{1}{2}+i\right)t} \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } e^{\left(-\frac{1}{2}-i\right)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

To write them in terms of trigonometric functions

$$x(t) = e^{-\frac{t}{2}} \begin{pmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{pmatrix},$$

$$\bar{x}(t) = e^{-\frac{t}{2}} \begin{pmatrix} \cos t - i \sin t \\ -i \cos t - \sin t \end{pmatrix} = \begin{pmatrix} e^{-\frac{t}{2}} \cos t \\ -e^{-\frac{t}{2}} \sin t \end{pmatrix} + i \begin{pmatrix} e^{-\frac{t}{2}} \sin t \\ e^{-\frac{t}{2}} \cos t \end{pmatrix}$$

Then the real and imaginary parts of $x(t)$ are also solutions.

Let $x_1 = e^{-\frac{t}{2}} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$, $x_2 = e^{-\frac{t}{2}} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$.

To check it they are linearly independent, we have

$$W[x_1, x_2](t) = \begin{vmatrix} e^{-\frac{t}{2}} \cos t & e^{-\frac{t}{2}} \sin t \\ -e^{-\frac{t}{2}} \sin t & e^{-\frac{t}{2}} \cos t \end{vmatrix} = e^{-t} \neq 0$$

$\therefore$ To generate real-valued solutions, we just write

$$x(t) = e^{-\frac{t}{2}} \begin{bmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{bmatrix} = c_1 x_1 + c_2 x_2$$

Q： What about the stability of the fixed point $x = 0$ as t increases ?

as $t \to \infty$, all solutions go to zero !

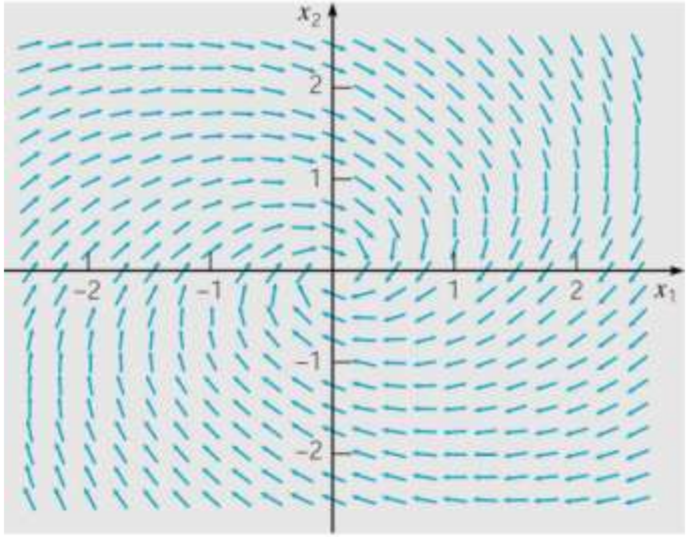
Example 1: Direction Field (1 of 7)

- Consider the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\mathbf{x}' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}\mathbf{x}$$

- A direction field for this system is given below.
- Substituting $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ in for $\mathbf{x}$, and rewriting system as $(\mathbf{A}\text{-}r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, we obtain

$$\begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

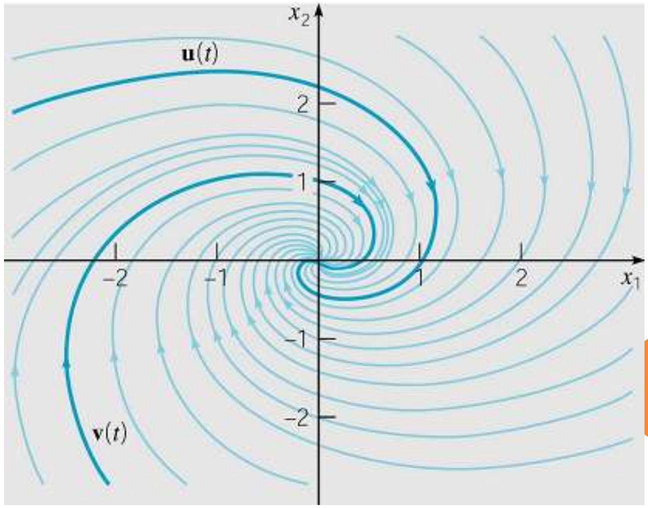


Example 1: Phase Plane (6 of 7)

- Given below is the phase plane plot for solutions **x**, with

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix}$$

- Each solution trajectory approaches origin along a spiral path as $t \to \infty$, since coordinates are products of decaying exponential and sine or cosine factors.
- The graph of **u** passes through (1,0), since u(0) = (1,0). Similarly, the graph of **v** passes through (0,1).
- The origin is a **spiral point**, and is asymptotically stable.

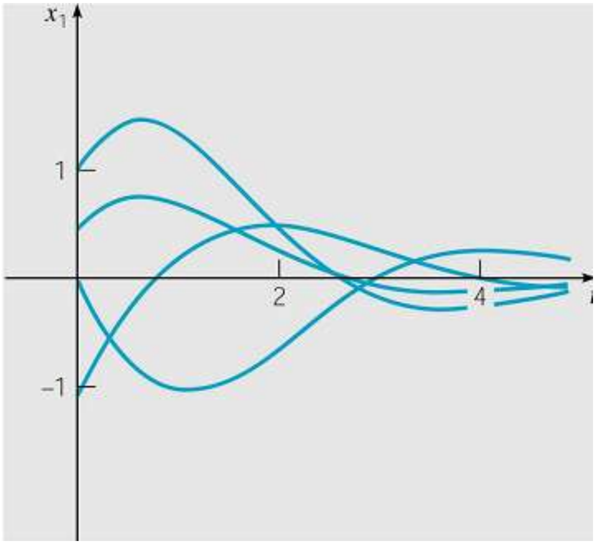


Example 1: Time Plots (7 of 7)

- The general solution is $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$:

$$\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t \\ -c_1 e^{-t/2}\sin t + c_2 e^{-t/2}\cos t \end{pmatrix}$$

- As an alternative to phase plane plots, we can graph x_1 or x_2 as a function of t. A few plots of x_1 are given below, each one a decaying oscillation as $t \to \infty$.



Real-Valued Solutions

- Thus for complex conjugate eigenvalues r_1 and r_2, the corresponding solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are conjugates also.
- To obtain real-valued solutions, use real and imaginary parts of either $\mathbf{x}^{(1)}$ or $\mathbf{x}^{(2)}$. To see this, let $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$. Then

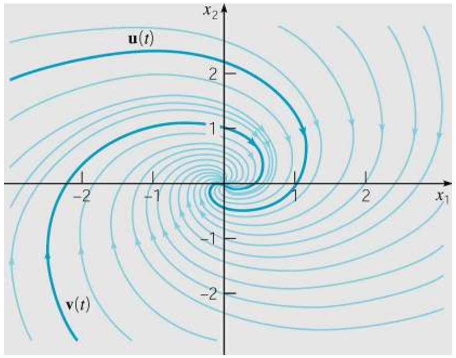
$$\begin{aligned}\mathbf{x}^{(1)} &= \boldsymbol{\xi}^{(1)} e^{(\lambda+i\mu)t} = (\mathbf{a}+i\mathbf{b})e^{\lambda t}(\cos\mu t + i\sin\mu t) \\ &= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t) \\ &= \mathbf{u}(t) + i\,\mathbf{v}(t)\end{aligned}$$

where

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t),\ \mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t),$$

are real valued solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and can be shown to be linearly independent.

Spiral Points, Centers, Eigenvalues, and Trajectories



- In previous example, general solution was

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} e^{-t/2} \cos t \\ -e^{-t/2} \sin t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t/2} \sin t \\ e^{-t/2} \cos t \end{pmatrix}$$

- The origin was a **spiral point**, and was asymptotically stable.
- If real part of complex eigenvalues is positive, then trajectories spiral away, unbounded, from origin, and hence origin would be an unstable spiral point.
- If real part of complex eigenvalues is zero, then trajectories circle origin, neither approaching nor departing. Then origin is called a **center** and is stable, but not asymptotically stable. Trajectories periodic in time.
- The direction of trajectory motion depends on entries in **A**.

7.7 Fundamental matrices

We consider the following linear system

$$\dot{x} = p(t)x$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & \cdots & \cdots & p_{2n} \\ \vdots & & & \vdots \\ p_{n1} & \cdots & \cdots & p_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Q： What is a fundamental matrix？

Suppose $x^{(1)}(\mathrm{t})$, $x^{(2)}(\mathrm{t})$, $\cdots$, $x^{(\mathrm{n})}(\mathrm{t})$ from a fundamental set of solutions, where

$$x^{(i)}(t) = \begin{pmatrix} x_1^{(i)}(t) \\ x_2^{(i)}(t) \\ \vdots \\ x_n^{(i)}(t) \end{pmatrix}$$

Then the matrix $\left(x^{(1)}(\mathrm{t}), x^{(2)}(\mathrm{t}), \cdots, x^{(\mathrm{n})}(\mathrm{t})\right)$

$$= \begin{pmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) & \cdots & x_1^{(n)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) & \cdots & x_2^{(n)}(t) \\ \vdots & \vdots & & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix} = \psi(t)$$

is called a fundamental matrix $\psi(t)$.

Remark：

A fundamental matrix is not unique！

＊ Claim：

Any fundamental matrix satisfies the matrix equation

$$\dot{\psi}(t) = p(t)\psi(t)\,.$$

Let us use a 2×2 matrix to explain.

Suppose $x^{(1)}$ & $x^{(2)}$ are fundamental solutions

$$x^{(1)}(t) = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix}$$

$$\frac{d}{dt}x^{(1)} = \begin{pmatrix} \dot{x}_1^{(1)} \\ \dot{x}_2^{(1)} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} \dot{x}_1^{(1)} &= p_{11}x_1^{(1)} + p_{12}x_2^{(1)} \\ \dot{x}_2^{(1)} &= p_{21}x_1^{(1)} + p_{22}x_2^{(1)} \end{aligned}$$

Similarly,

$$\begin{aligned} \dot{x}_1^{(2)} &= p_{11}x_1^{(2)} + p_{12}x \\ \dot{x}_2^{(2)} &= p_{21}x_1^{(2)} + p_{22}x \end{aligned}$$

$\therefore$ We can write it as a matrix equation

$$\frac{d}{dt}\begin{pmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{pmatrix}$$

Ex： Find a fundamental matrix for the system

$$\dot{x} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x$$

sol： To solve the linear system, we solve the eigenvalues and eigenvectors.

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 + 4 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

$$\therefore \ \lambda = 2 \ \ or \ \ -1$$

- $$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \ 3v_1 - 2v_2 = 2v_1$$

$$\Rightarrow \ v_1 = 2v_2 \qquad \therefore \ v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

- $$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \quad 3v_1 - 2v_2 = -v_1$$

$$\Rightarrow \quad 4v_1 = 2v_2 \qquad\qquad \therefore \quad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\therefore$ Two independent solutions are $\begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix}$ and $\begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix}$

$\therefore$ The fundamental matrix can be written as

$$\psi(t) = \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix}$$

Verify ψ satisfies $\dot{\psi} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \psi$

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix} = \begin{pmatrix} 4e^{2t} & -e^{-t} \\ 2e^{2t} & -2e^{-t} \end{pmatrix} = \dot{\psi}$$

$\therefore$ $\psi(t)$ indeed is a fundamental matrix !

Note that any independent solutions can form fundamental solutions, to make it unique, we need to specify additional condition.

Now we introduce a special one

$$\begin{cases} \dot{\Phi} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \Phi \\ \Phi(0) = \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

The general form of a fundamental solution is

$$\begin{pmatrix} 2c_1e^{2t}+c_2e^{-t} & 2c_3e^{2t}+c_4e^{-t} \\ c_1e^{2t}+2c_2e^{-t} & c_3e^{2t}+2c_4e^{-t} \end{pmatrix}_{t=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 2c_1+c_2 & 2c_3+c_4 \\ c_1+2c_2 & c_3+2c_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad c_1=\frac{2}{3}, \quad c_2=-\frac{1}{3}, \quad c_3=-\frac{1}{3}, \quad c_4=\frac{2}{3}$$

$$\therefore \quad \Phi(t)=\begin{pmatrix} \frac{4}{3}e^{2t}-\frac{1}{3}e^{-t} & -\frac{2}{3}e^{2t}+\frac{2}{3}e^{-t} \\ \frac{2}{3}e^{2t}-\frac{2}{3}e^{-t} & -\frac{1}{3}e^{2t}+\frac{4}{3}c_4e^{-t} \end{pmatrix}$$

$\Phi(t)$ is just a fundamental matrix with a specific initial condition.
We can also specify the initial condition at $t = t_0$.

$\therefore$ In general, $\Phi(t)$ satisfies $\begin{cases} \dot{\Phi}(t) = p(t)\Phi(t) \\ \Phi(t_0) = \mathrm{I} \end{cases}$

Q： What's the use of $\Phi(t)$?

Consider the following initial value problem

$\begin{cases} \dot{x}(t) = p(t)x \\ x(t_0) = x_0 \end{cases}$, then $x(t) = \Phi(t)x_0$

where $\Phi(t)$ satisfies $\begin{cases} \dot{\Phi}(t) = p(t)\Phi(t) \\ \Phi(t_0) = \mathrm{I} \end{cases}$

pf： Let $x(t)=\Phi(t)x_0$

$$\Rightarrow \quad \dot{x}(t)=\dot{\Phi}(t)x_0=p(t)\Phi(t)x_0=p(t)x$$

Moreover, $x(t_0)=\Phi(t_0)x_0=\mathrm{I}\cdot x_0=x_0$

$\therefore$ $x(t)$ solves the problem.

$\therefore$ If we find $\Phi(t)$, we can solve $x(t)$ in terms of $\Phi(t)$.

- There are three ways to compute $\Phi(t)$.

(1) We have computed the first method.

(2) Let $x^{(i)}(t)s$ be fundamental solutions of $\dot{x} = p(t)x$.

Let $\psi(t) = \left(x^{(1)}(\mathrm{t}),\ x^{(2)}(\mathrm{t}), \cdots,\ x^{(n)}(\mathrm{t})\right)$, then any solution of $\dot{x} = p(t)x(t)$

can be written by $x(t) = c_1 x^{(1)}(\mathrm{t}) + c_2 x^{(2)}(\mathrm{t}) + \cdots + c_n x^{(n)}(\mathrm{t})$

$$= \left[x^{(1)},\ x^{(2)}, \cdots,\ x^{(n)}\right] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \psi(t) \cdot c \qquad \text{where } c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\because \quad x(t_0) = \psi(t_0) \cdot c = x_0 \quad \Rightarrow \quad c = \psi(t_0)^{-1} x_0$$

$$\therefore \quad x(t) = \psi(t) \cdot c = \psi(t)\psi(t_0)^{-1} x_0 = \Phi(t) x_0$$

$$\therefore \quad \Phi(t) = \psi(t)\psi(t_0)^{-1}$$

In case $t_0 = 0$, $\Phi(t) = \psi(t)\psi(0)^{-1}$.

Hence we know how to compute $\Phi(t)$ from $\psi(t)$.

Ex： $\dot{x}=\begin{pmatrix}3 & -2\\ 2 & -2\end{pmatrix}x$

Find $\Phi(t)$ such that $\Phi(0)=\mathrm{I}$.

sol： From the formula, $\Phi(t)=\psi(t)\psi(0)^{-1}$.

$\therefore$ First find a fundamental matrix $\psi(t)$, which we have done earlier.

$\psi(t)=\begin{pmatrix}2e^{2t} & e^{-t}\\ e^{2t} & 2e^{-t}\end{pmatrix}$ Then $\psi(0)=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$, $\psi^{-1}(0)=\frac{1}{3}\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$

$$\Phi(t)=\begin{pmatrix}2e^{2t} & e^{-t}\\ e^{2t} & 2e^{-t}\end{pmatrix}=\begin{pmatrix}\frac{2}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3}\end{pmatrix}=\begin{pmatrix}\frac{4}{3}e^{2t}-\frac{1}{3}e^{-t} & -\frac{2}{3}e^{2t}+\frac{2}{3}e^{-t}\\ \frac{2}{3}e^{2t}-\frac{2}{3}e^{-t} & -\frac{1}{3}e^{2t}+\frac{4}{3}c_4e^{-t}\end{pmatrix}$$

(3) An alternative way to find $\Phi(t)$

Motivation： Consider a scalar initial value problem

$$\begin{cases} \dot{x} = a\,x \\ x(0) = x_0 \end{cases} \Rightarrow \quad x = e^{at} x_0$$

- For a system $\begin{cases} \dot{\mathrm{x}} = \mathrm{A}\,\mathrm{x} \\ \mathrm{x}(0) = x^0 \end{cases}$ A ： $n \times n$ constant matrix

$$\Rightarrow \quad \mathrm{x} = \Phi(t) x^0 \quad \text{where} \quad \begin{cases} \dot{\Phi} = \mathrm{A}\Phi \\ \Phi(0) = \mathrm{I} \end{cases}$$

Q： Can we write $\mathrm{x}(\mathrm{t})$ in exponential form as we did for a scalar ODE ?

i.e. , can we write $e^{\mathrm{A}t}\mathrm{x}^0 = \mathrm{x}(t)$?

How do we define $e^{\mathrm{A}t}$ for a matrix A ?

* The matrix $\exp(\mathrm{A}t) = e^{\mathrm{A}t}$

For the scalar e^{at}, we can use the power series

$$e^{at} = 1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots + \frac{1}{n!}a^nt^n + \cdots$$

Now we generalize this idea for a matrix A

$$e^{\mathrm{A}t} = \mathrm{I} + \mathrm{A}t + \frac{1}{2!}\mathrm{A}^2t^2 + \frac{1}{3!}\mathrm{A}^3t^3 + \cdots + \frac{1}{n!}\mathrm{A}^nt^n + \cdots$$

If we differentiate the series term by term, we obtain

$$\frac{d}{dt}\left(e^{\mathrm{A}t}\right) = \mathrm{A} + \mathrm{A}^2t + \frac{1}{2!}\mathrm{A}^3t^2 + \cdots + \frac{1}{(n-1)!}\mathrm{A}^nt^{n-1} + \cdots$$

$$= \mathrm{A}\left[\mathrm{I} + \mathrm{A}t + \frac{1}{2!}\mathrm{A}^2t^2 + \cdots + \frac{1}{(n-1)!}\mathrm{A}^{n-1}t^{n-1} + \cdots\right]$$

$$= \mathrm{A}e^{\mathrm{A}t}$$

Moreover, at $t = 0$, $e^{\mathrm{A}t} = \mathrm{I}$.

$\therefore$ $e^{\mathrm{A}t}$ satisfies $\begin{cases} \dot{\Phi}(t) = \mathrm{A}\Phi \\ \Phi(0) = \mathrm{I} \end{cases} \Rightarrow \begin{matrix} e^{\mathrm{A}t} = \Phi(t) \\ \textit{with } \Phi(0) = \mathrm{I} \end{matrix}$

Q： How to compute $e^{\mathrm{A}t}$ from the power series ？

A： By the Jordan form of A

Suppose $\mathrm{A} = \mathrm{T}J\mathrm{T}^{-1}$

$$\Rightarrow \quad \mathrm{A}^2 = \mathrm{T}J\mathrm{T}^{-1}\mathrm{T}J\mathrm{T}^{-1} = \mathrm{T}J^2\mathrm{T}^{-1}$$

$$\vdots$$

$$\mathrm{A}^n = \mathrm{T}J^n\mathrm{T}^{-1}$$

$$\therefore \quad e^{\mathrm{A}t} = \mathrm{I} + \mathrm{A}t + \frac{1}{2!}\mathrm{A}^2t^2 + \cdots + \frac{1}{n!}\mathrm{A}^nt^n + \cdots$$

$$= \mathrm{I} + \mathrm{T}J\mathrm{T}^{-1}t + \frac{1}{2!}\mathrm{T}J^2\mathrm{T}^{-1}t^2 + \cdots + \frac{1}{n!}\mathrm{T}J^n\mathrm{T}^{-1}t^n + \cdots$$

$$= \mathrm{T}\left[\mathrm{I} + Jt + \frac{1}{2!}J^2t^2 + \cdots + \frac{1}{n!}J^nt^n + \cdots\right]\mathrm{T}^{-1}$$

$$= \mathrm{T}e^{Jt}\mathrm{T}^{-1}$$

For diagonalizable matrix A (2×2)

$$A = T\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}T^{-1} \qquad \therefore J = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

$$\Rightarrow \quad e^{Jt} = I + Jt + \frac{1}{2!}J^2t^2 + \cdots + \frac{1}{n!}J^nt^n + \cdots$$

$$= I + \begin{bmatrix}\lambda_1 t & 0\\ 0 & \lambda_2 t\end{bmatrix} + \frac{1}{2!}\begin{bmatrix}\lambda_1^2t^2 & 0\\ 0 & \lambda_2^2t^2\end{bmatrix} + \cdots + \frac{1}{n!}\begin{bmatrix}\lambda_1^nt^n & 0\\ 0 & \lambda_2^nt^n\end{bmatrix} + \cdots$$

$$= \begin{bmatrix}1+\lambda_1 t + \frac{1}{2!}\lambda_1^2t^2 + \cdots + \frac{1}{n!}\lambda_1^nt^n + \cdots & 0\\ 0 & 1+\lambda_2 t + \frac{1}{2!}\lambda_2^2t^2 + \cdots + \frac{1}{n!}\lambda_2^nt^n + \cdots\end{bmatrix}$$

$$= \begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}$$

- $n \times n$ diagonalizable matrix A
- The final method to solve $\dot{x} = \mathrm{A}x$

$$\mathrm{A} = \mathrm{T}J\mathrm{T}^{-1} \qquad \therefore \quad \dot{x} = \mathrm{T}J\mathrm{T}^{-1}x$$

$$\Rightarrow \quad \mathrm{T}^{-1}\dot{x} = J\mathrm{T}^{-1}x \quad \Rightarrow \quad \dot{y} = Jy$$

Ex： $\mathrm{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

$$\lambda = 2 \ , \quad \xi^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \lambda = -1 \ , \quad \xi^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad \mathrm{T} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} , \quad \mathrm{T}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = D = \mathrm{T}^{-1}\mathrm{A}\mathrm{T} \qquad \therefore \quad \mathrm{A} = \mathrm{T}D\mathrm{T}^{-1}$$

$$e^{\mathrm{A}t} = \mathrm{T}e^{Dt}\mathrm{T}^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} & -\frac{2}{3}e^{2t} + \frac{2}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t} \end{pmatrix}$$

Q： What's the use of e^{At} ?

If we know the initial data is x_0, then $x(t) = e^{At} x_0$.
Besides, e^{At} is useful for the nonhomogeneous problem.