

## 7.4 Basic theory of systems of first order linear equations

- \*  $\dot{x} = p(t)x + g(t)$  (vector equation)
- \* Fundamental set of solutions
- \* Concept of Wronskian

- We consider the following system of n first order linear equations

$$\begin{aligned}\dot{x}_1 &= p_{11}(t)x_1 + \dots + p_{1n}(t)x_n + g_1(t) \\ &\vdots \\ \dot{x}_n &= p_{n1}(t)x_1 + \dots + p_{nn}(t)x_n + g_n(t)\end{aligned}$$

- If we let  $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ,  $p(t) = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}$ ,  $g(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$ ,

then we can write the system in matrix notation  $\dot{x} = p(t)x + g(t)$ .



- The advantage to write the equations in matrix notation is that we can use the theory from linear algebra.

- A vector  $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \phi(t) = \begin{pmatrix} \phi_1(t) \\ \vdots \\ \phi_n(t) \end{pmatrix}$  is said to be a solution

of the system  $\dot{x} = p(t)x + g(t)$  if its components satisfy the system of equations.

- assumptions :

$p_{11}, \dots, p_{nn}, g_1, \dots, g_n$  are continuous on  $\alpha < t < \beta$ , then by Theorem 7.1.2, there is a unique solution to the system on  $\alpha < t < \beta$ .



- Now we first consider the homogeneous equation

$$\dot{x} = p(t)x . \quad *$$

- Let  $x^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \dots, x^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}$

- We have the following theorem :  
(Principle of superposition)



# Theorem 7.4.1

(Principle of superposition)

- If the vector functions  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$  are solutions of the system  $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ , then the linear combination  $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$  is also a solution for any constants  $c_1$  and  $c_2$ .

- Note: By repeatedly applying the result of this theorem, it can be seen that every finite linear combination

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \dots + c_k\mathbf{x}^{(k)}(t)$$

of solutions  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)}$  is itself a solution to  $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ .



Q : as before, can all solutions be found in this way ?

Is it sufficient to find  $n$  solutions ?

Are they linearly independent ?

Here comes in the concept of Wronskian.

- Let  $x^{(1)}, \dots, x^{(n)}$  be  $n$  solutions of the  $n$ th order system  $\dot{x} = p(t)x$ ,

$$\text{where } x^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}.$$



- Consider the matrix  $\underline{\bar{x}}(t) = (x^{(1)}(t), x^{(2)}(t), \dots, x^{(n)}(t))$

$$= \begin{bmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{bmatrix}$$

then,  $x^{(1)}, \dots, x^{(n)}$  are linearly independent iff  $\det \underline{\bar{x}}(t) \neq 0$ .

**Def :** The Wronskian of the  $n$  solutions  $x^{(1)}, \dots, x^{(n)}$  is defined by

$$W[x^{(1)}, \dots, x^{(n)}](t) = \det \underline{\bar{x}}(t)$$



## Theorem 7.4.2

- If  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$  are linearly independent solutions of the system  $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$  for each point in  $I$ :  $\alpha < t < \beta$ , then each solution  $\mathbf{x} = \boldsymbol{\phi}(t)$  can be expressed uniquely in the form

$$\mathbf{x} = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) + \dots + c_n \mathbf{x}^{(n)}(t)$$

- If solutions  $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$  are linearly independent for each point in  $I$ :  $\alpha < t < \beta$ , then they are **fundamental solutions on  $I$** , and the **general solution** is given by

$$\mathbf{x} = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) + \dots + c_n \mathbf{x}^{(n)}(t)$$



pf : Let  $x = \phi(t)$  be any solution of  $\dot{x} = p(t)x$   
with initial condition  $x(t_0) = \xi$

We want to show that  $x$  can be written as  
a linear combination of  $x^{(1)}, x^{(2)}, \dots, x^{(n)}$ ,  
i.e. we need to solve  $c_1, c_2, \dots, c_n$  such that

$$x(t) = c_1 x^{(1)}(t) + \dots + c_n x^{(n)}(t).$$

since  $x^{(1)}, \dots, x^{(n)}$  are linearly independent,

$$\det \begin{bmatrix} x^{(1)} & x^{(2)} & \dots & x^{(n)} \end{bmatrix} \neq 0 \quad \forall t.$$

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Of course it satisfies the differential equation.

We want to determine  $c_1, c_2, \dots, c_n$  such that the initial condition is satisfied.

To solve  $x(t_0) = \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$ , we write it as scalar equations

$$\begin{aligned} c_1 x_{11}(t_0) + \dots + c_n x_{1n}(t_0) &= \xi_1 \\ c_1 x_{21}(t_0) + \dots + c_n x_{2n}(t_0) &= \xi_2 \\ &\vdots \\ c_1 x_{n1}(t_0) + \dots + c_n x_{nn}(t_0) &= \xi_n \end{aligned},$$

clearly,  $c_1, \dots, c_n$  are uniquely solved

since  $x^{(1)}, \dots, x^{(n)}$  are fundamental solutions. ///



## Theorem 7.4.3

- If  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$  are solutions of the system  $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$  on  $I: \alpha < t < \beta$ , then the Wronskian  $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}](t)$  is either identically zero on  $I$  or else is never zero on  $I$ .

pf : The idea is to derive an ODEs for  $W(t)$ .

$$\begin{aligned} \frac{dW}{dt} &= (p_{11} + p_{22} + \dots + p_{nn})W \quad * \\ &= \text{tr}(\mathbf{p}) W. \end{aligned}$$

$$\Rightarrow \int \frac{dW}{dt} = \int \text{tr}(\mathbf{p}) dt$$

$$\Rightarrow \ln |W| = \int \text{tr}(\mathbf{p}) dt$$

$$\Rightarrow W = c e^{\int \text{tr}(\mathbf{p}) dt}$$

$$\therefore W(t) \equiv 0 \quad \text{or} \quad W \neq 0 \quad \forall \alpha < t < \beta$$

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Let us look at  $*$  for  $2 \times 2$  matrices :

$$W(t) = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

$$(\therefore \frac{dW}{dt} = \begin{vmatrix} \dot{x}_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & & \vdots \\ \dot{x}_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} + \begin{vmatrix} x_{11} & \dot{x}_{12} & \cdots & x_{1n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & \dot{x}_{n2} & \cdots & x_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} x_{11} & \cdots & \dot{x}_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & \dot{x}_{nn} \end{vmatrix})$$

From the ODE  $\dot{x} = p(t)x$

$$\Rightarrow \begin{pmatrix} \dot{x}_{11} \\ \dot{x}_{21} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} \quad \& \quad \begin{pmatrix} \dot{x}_{12} \\ \dot{x}_{22} \end{pmatrix} = \begin{pmatrix} p_{11}x_{12} + p_{12}x_{22} \\ p_{21}x_{12} + p_{22}x_{22} \end{pmatrix}$$

$$\therefore \frac{dW}{dt} = \begin{vmatrix} p_{11}x_{11} + p_{12}x_{21} & p_{11}x_{12} + p_{12}x_{22} \\ p_{21}x_{11} + p_{22}x_{21} & p_{21}x_{12} + p_{22}x_{22} \end{vmatrix} = (p_{11} + p_{22})[x_{11}x_{22} - x_{12}x_{21}]$$

$$= \text{tr}(p) \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = \text{tr}(p) W \quad ///$$

