Chapter 9

Nonlinear differential equations and stability

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9.1 The phase plane： Linear system

Many differential equations can not be solved analytically.
How can we get some qualitative information of the solution without solving the equation itself？
First we introduce the concepts of critical points and stability.

Def： Autonomous system $\dot{x} = f(x)$

non-autonomous system $\dot{x} = f(x,t)$

Def： The points where $f(x)=0$ are called **critical points** (**equilibrium solutions**) of the autonomous system.

Def：

1) A critical point x_0 is said to be **stable** if $\forall \varepsilon > 0,\ \exists \delta > 0$ ∍ every solution $x=\varphi(t)$ satisfying $\|\varphi(0)-x^0\| < \delta$ exists for all $t>0$ and $\|\varphi(t)-x_0\| < \varepsilon \quad \forall t \ge 0$.

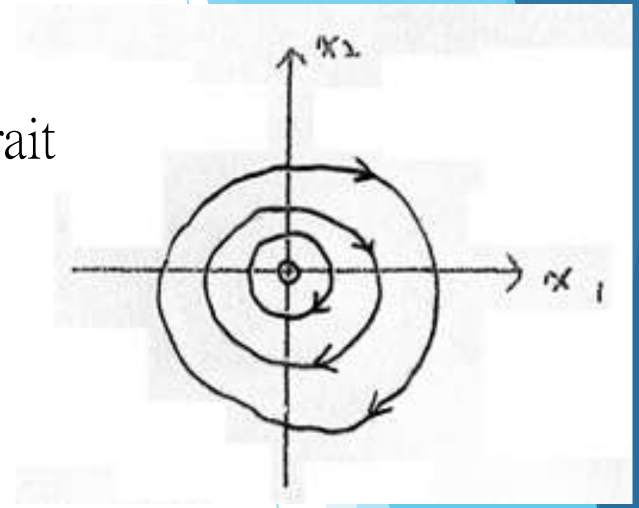
2) A critical point that is not stable is said to be **unstable.**

3) A critical point x_0 is said to be **asymptotically stable** if it is stable and $\lim_{t\to\infty} \varphi(t) = x^0$

- In this section, we deal with the stability issue of the 2×2 linear system.
- Consider $\dot{x} = Ax$ where A is a 2×2 matrix and nonsingular.
  Then the only critical point (equilibrium solution) is $x = 0$.
- A vector solution $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \phi(t)$ of $\dot{x} = Ax$ can be considered as a **parametric representation** for a curve in the (x_1, x_2) plane.
  This curve is regarded as the path, or trajectory of a moving particle whose velocity $\dfrac{dx}{dt}$ is specified by the differential equation.
- The (x_1, x_2) plane is called the **phase plane** and a representative set of trajectories is referred to a phase portrait.

Ex： $\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$

phase portrait



↑ phase plane

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned} \Rightarrow \ddot{x}_1 = \dot{x}_2 = -x_1 \Rightarrow \begin{aligned} x_1 &= c_1 \sin t + c_2 \cos t \\ x_2 &= c_1 \cos t - c_2 \sin t \end{aligned}$$

$\therefore \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \phi(t) = \begin{pmatrix} c_1 \sin t + c_2 \cos t \\ c_1 \cos t - c_2 \sin t \end{pmatrix}$ is a solution of the ODE.

What is the curve described by the parametric representation ?

$x_1{}^2 + x_2{}^2 = c_1{}^2 + c_2{}^2 = c$, $\therefore$ the trajectory is a circle for different value of c.

Now we will discuss these issues in details according to the signs of the eigenvalues.

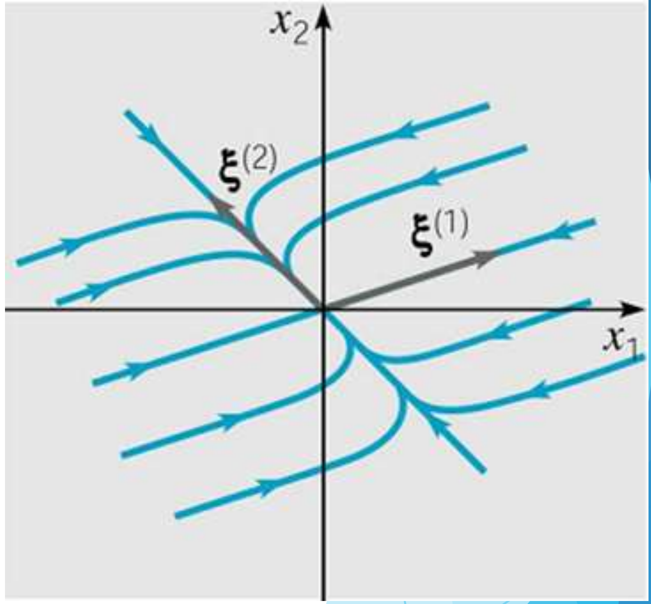
Case 1： Real unequal eigenvalues of the some sign.

Let the solution of $\dot{x} = \mathrm{A}x$ be $x(t) = c_1\xi^{(1)}e^{r_1 t} + c_2\xi^{(2)}e^{r_2 t}$

where r_1 and r_2 are eigenvalues of A.

Suppose r_1, r_2 are both negative.

Regardless of the values of c_1 and c_2, all solutions tend to zero as $t \to \infty$



∴ The critical point 0 is **asymptotically stable.**

- If we look into the details of the solutions, suppose $r_1 < r_2 < 0$,

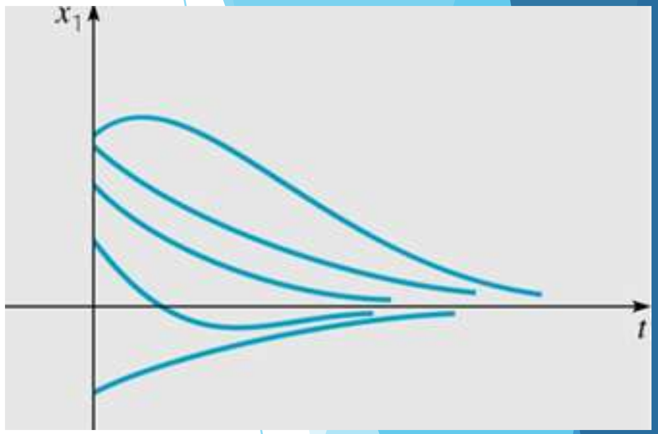
∴ the solution can be written as

$$x(t) = c_1\xi^{(1)}e^{r_1 t} + c_2\xi^{(2)}e^{r_2 t} \quad x(t) = c_1\xi^{(1)}e^{r_1 t} + c_2\xi^{(2)}e^{r_2 t}$$

From the expression, we can see that as $t \to \infty$, the first term $c_1\xi^{(1)}e^{(r_1-r_2)t}$ is negligible compared to $c_2\xi^{(2)}$.

$\therefore$ as time is big, all the solutions tend to zero in the direction of $\xi^{(2)}$, r_2 is the dominant eigenvalue.

This type of critical point is called a **node** or **nodal sink.**



- If r_1 and r_2 are both positive and $0 < r_2 < r_1$, then the solution has the same pattern.

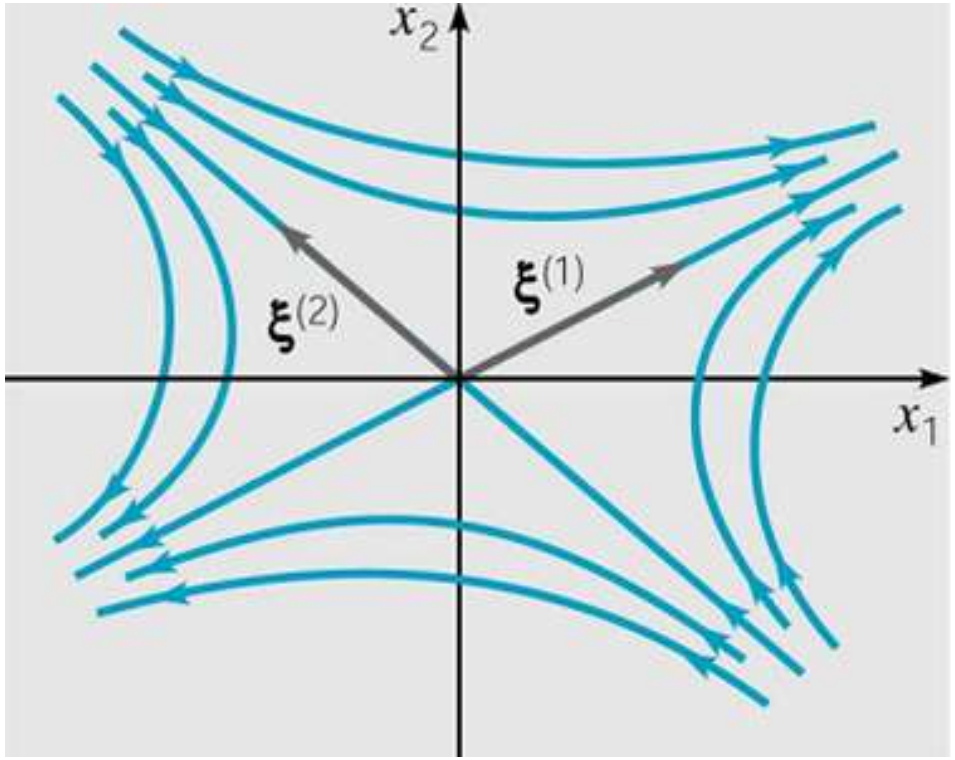
The only difference is that the direction is just the opposite.

In this case, the critical point is called a **nodal source.**

$$x(t) = e^{r_2 t}\left(c_1\xi^{(1)}e^{(r_1-r_2)t} + c_2\xi^{(2)}\right)$$

Case 2： Real eigenvalues of opposite sign

$x(t) = c_1\xi^{(1)}e^{r_1 t} + c_2\xi^{(2)}e^{r_2 t}$ where $r_1 > 0$ and $r_2 < 0$.



The critical point is called a **saddle point.**

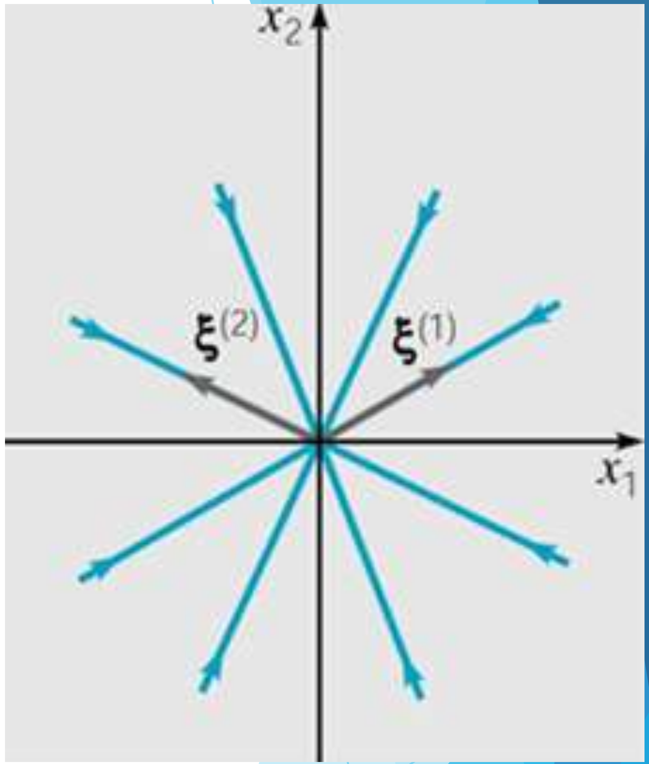
Case 3： Equal eigenvalues

Now $r_1 = r_2 = r$.

We may suppose $r < 0$.

For the other case, we only need to reverse the direction on the trajectory.

A) There are two independent eigenvectors.



$$x(t) = c_1\xi^{(1)}e^{rt} + c_2\xi^{(2)}e^{rt} \quad , \quad r < 0$$

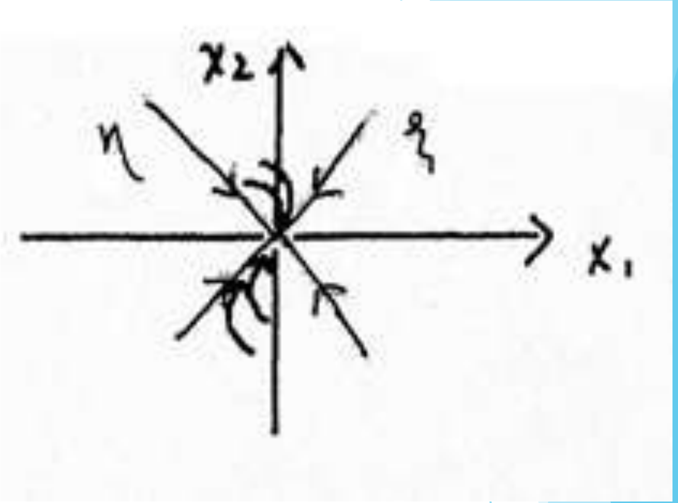
$$= e^{rt}\left(c_1\xi^{(1)} + c_2\xi^{(2)}\right)$$

The critical point is called a **proper node** or **star point.**

B) There is only one eigenvector ξ.

Recall from before $x(t) = c_1\xi e^{rt} + c_2\xi t e^{rt} + c_2\eta e^{rt}$

where η is the generalized eigenvector.



Suppose $r < 0$, then as $t \to \infty$, no matter what c_1 and c_2 are, the solution will go to zero.

$$x(t) = e^{rt}\left(c_1\xi + c_2\xi t + c_2\eta\right)$$

We can see as t large, $c_2\xi t$ is the dominant term compared with $c_1\xi + c_2\eta$.

$\therefore$ as t goes to ∞, the solution will go to zero in the direction of ξ.

For $r_1 = r_2 > 0$, it is similar to draw the picture.

We just need to reverse the direction in t variable.

In the presence of one independence eigenvector, the critical point 0 is called an **improper** or **degenerate node**.

Case 4： Complex eigenvalues

In this case, suppose the eigenvalues are $\lambda + i\mu$ and $\lambda - i\mu$,

and the corresponding eigenvectors are ξ and η $\Rightarrow$ $\eta = \bar{\xi}$

$\therefore$ One solution can be written as

$$x(t) = \xi e^{(\lambda + i\mu)t} = e^{\lambda t}\begin{pmatrix} \xi_1(\cos\mu t + i\sin\mu t) \\ \xi_2(\cos\mu t + i\sin\mu t) \end{pmatrix} \qquad \bar{x}(t) = e^{\lambda t}\begin{pmatrix} \bar{\xi}_1(\cos\mu t - i\sin\mu t) \\ \bar{\xi}_2(\cos\mu t - i\sin\mu t) \end{pmatrix}$$

$$\Rightarrow \quad x + \bar{x} = e^{\lambda t}\begin{pmatrix} 2\,\mathrm{Re}\,\xi_1 \cos\mu t - 2\,\mathrm{Im}\,\xi_1 \sin\mu t \\ 2\,\mathrm{Re}\,\xi_2 \cos\mu t - 2\,\mathrm{Im}\,\xi_2 \sin\mu t \end{pmatrix}$$

$$x - \bar{x} = e^{\lambda t}\begin{pmatrix} 2i\,\mathrm{Im}\,\xi_1 \cos\mu t + 2i\,\mathrm{Re}\,\xi_1 \sin\mu t \\ 2i\,\mathrm{Im}\,\xi_2 \cos\mu t + 2i\,\mathrm{Re}\,\xi_2 \sin\mu t \end{pmatrix}$$

$\therefore$ All solutions can be generated by

$$x(t) = c_1 \operatorname{Re}\xi e^{\lambda t}\cos\mu t - c_1 \operatorname{Im}\xi e^{\lambda t}\sin\mu t + c_2 \operatorname{Im}\xi e^{\lambda t}\cos\mu t + c_2 \operatorname{Re}\xi e^{\lambda t}\sin\mu t$$

$$= \left(c_1 \operatorname{Re}\xi + c_2 \operatorname{Im}\xi\right)e^{\lambda t}\cos\mu t + \left(c_2 \operatorname{Re}\xi - c_1 \operatorname{Im}\xi\right)e^{\lambda t}\sin\mu t$$

$$x = \begin{pmatrix} \left(c_1 \operatorname{Re}\xi_1 + c_2 \operatorname{Im}\xi_1\right)e^{\lambda t}\cos\mu t + \left(c_2 \operatorname{Re}\xi_1 - c_1 \operatorname{Im}\xi_1\right)e^{\lambda t}\sin\mu t \\ \left(c_1 \operatorname{Re}\xi_2 + c_2 \operatorname{Im}\xi_2\right)e^{\lambda t}\cos\mu t + \left(c_2 \operatorname{Re}\xi_2 - c_1 \operatorname{Im}\xi_2\right)e^{\lambda t}\sin\mu t \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

From the expression, we only know the solution will go to zero if $\lambda < 0$.

But it is difficult to analyze how it goes to zero in details.

We consider a special case：

$$\dot{x} = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} x = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

We see the eigenvalues of the matrix are $\lambda \pm i\mu$.

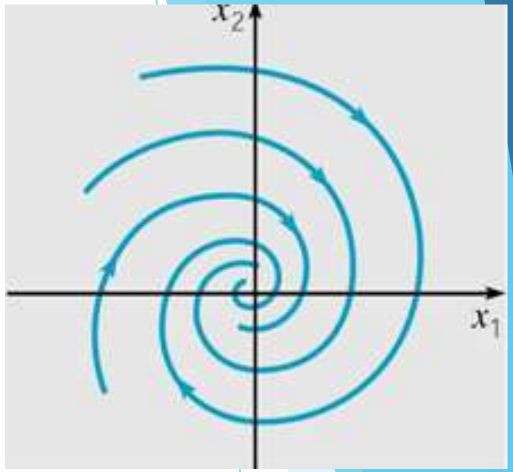
$$\begin{aligned} \dot{x}_1 &= \lambda x_1 + \mu x_2 \\ \dot{x}_2 &= -\mu x_1 + \lambda x_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}_1 x_1 &= \lambda x_1^2 + \mu x_1 x_2 \\ \dot{x}_2 x_2 &= -\mu x_1 x_2 + \lambda x_2^2 \end{aligned}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2\right) = \lambda\left(x_1^2 + x_2^2\right)$$

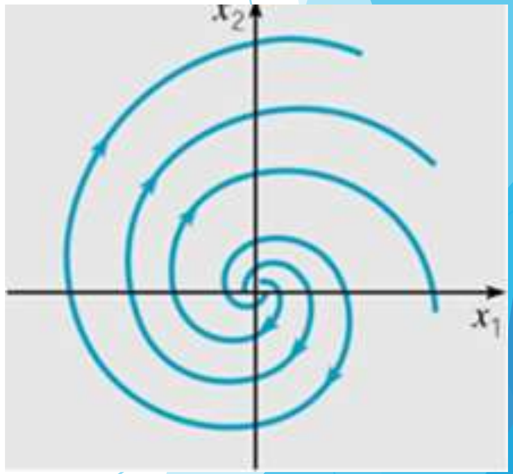
If $\lambda < 0$, we know the solution will go to zero as $t \to \infty$.

But the direction is unknown.

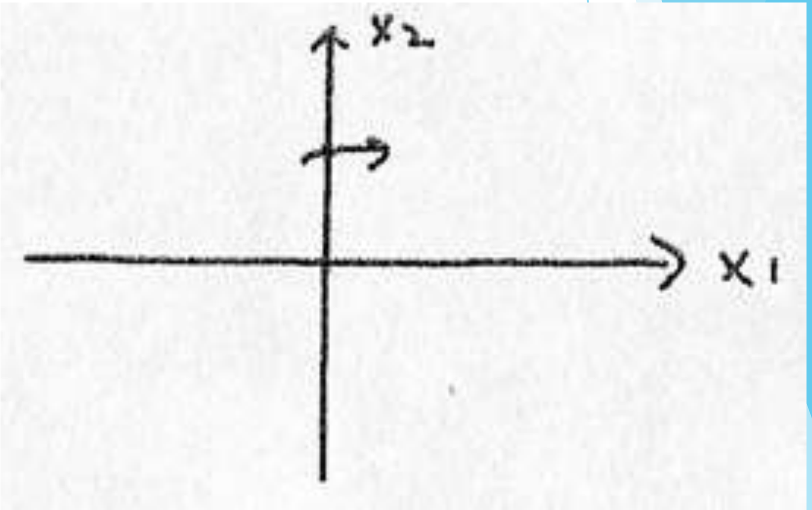
A quick way to know about this is to go back to the ODE $\dot{x}_1 = \lambda x_1 + \mu x_2$



or



If $\mu > 0$, then for the points on the positive $x_2 - axis$ $\dot{x}_1 > 0$.



∴ the point is moving clockwise.

∴ It belongs to the first category.

∴ For $\lambda < 0$, the critical point is a **spiral sink**；

$\lambda > 0$, it becomes a **spiral source.**

- Now we will introduce a useful method to determine the behavior near the critical point.
- It makes our life easier if we see the Polar coordinates.

Why？

- If we know the sign of $\dot{r}$, then we know whether the point goes to zero as ∞ since r denotes the distance to the critical point zero.
- The sign of $\dot{\theta}$ gives us information on the direction of the trajectory.

Q： How to use the Polar coordinates ?

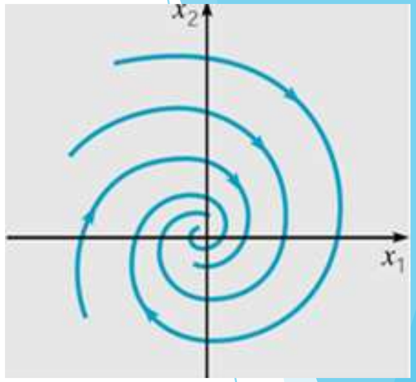
We need to derive equations for r and θ.

$r^2 = x_1^2 + x_2^2$, we already did this before and got $\dot{r} = \frac{dr}{dt} = \lambda r$.

To get an equation for θ, we notice that $\theta = \tan^{-1}\frac{x_2}{x_1}$.

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d}{dt}\left(\tan^{-1}\frac{x_2}{x_1}\right) = \frac{\frac{d}{dt}\left(\frac{x_2}{x_1}\right)}{1+\left(\frac{x_2}{x_1}\right)^2} = \frac{\frac{\dot{x}_2}{x_1} - \frac{\dot{x}_1 x_2}{x_1^2}}{1+\frac{x_2^2}{x_1^2}} = \frac{x_1\dot{x}_2 - \dot{x}_1 x_2}{x_1^2 + x_2^2}$$

$$= \frac{1}{r^2}\left[x_1(-\mu x_1 + \lambda x_2) - (\lambda x_1 + \mu x_2)x_2\right] = \frac{1}{r^2}\left[-\mu r^2\right] = -\mu$$



$\therefore$ If $\mu > 0$, then θ is decreasing,

that means the solution goes clockwise.

$\therefore$ From the equation of r, we know that if $\lambda > 0$,

r is increasing with t, so it is a **spiral source.**

If $\lambda < 0$, the critical point is a **spiral sink.**

What happens if $\lambda = 0$?

Case 5： Pure imaginary eigenvalues

In this case, we have two eigenvalues $\pm i\mu$.

From the equations in Polar coordinates.

$$\dot{r} = 0 \qquad \dot{\theta} = -\mu$$

$$\Rightarrow \quad r = \text{constant}, \quad \theta = -\mu t + \theta_0$$

We know the trajectories are concentric circles.

As before, the sign of μ determines the direction of the trajectory.

This kind of critical point is called the **center.**

It is stable, but not asymptotically stable.

To summarize：

Let r_1 and r_2 be the eigenvalues of $\mathbf{A}$.

Eigenvalues	**Type of Critical Point**	**Stability**
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically Stable
$r_2 < 0 < r_1$	Saddle Point	Unstable
$r_1 = r_2 > 0$	Proper or Improper Node	Unstable
$r_1 = r_2 < 0$	Proper or Improper Node	Asymptotically Stable
$r_1, r_2 = \lambda \pm i\mu$	Spiral Point	
$\lambda > 0$		Unstable
$\lambda < 0$		Asymptotically Stable
$r_1 = i\mu,\ r_2 = -i\mu$	Center	Stable

9.2 Autonomous systems and stability

We are concerned with systems of two simultaneous differential equations of the form $\begin{cases} \dfrac{dx}{dt} = F(x, y) \\ \dfrac{dy}{dt} = G(x, y) \end{cases}$ where F & G are continuous and have continuous partial derivatives in some domain D of the xy- plane.

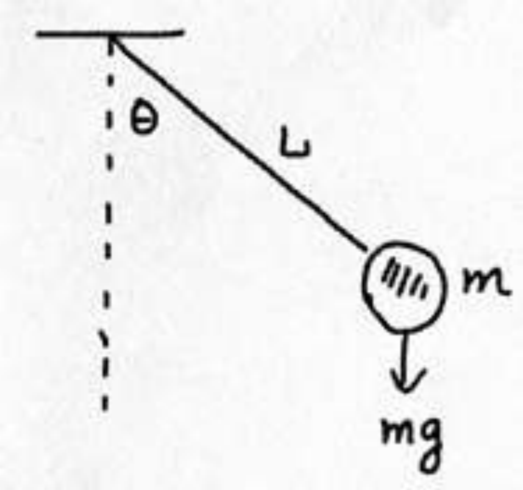
We observe that the functions F & G do not depend on the independent variable t, but only on the dependent variable x & y.

Such system is called **autonomous systems** occur frequently in applications, and the analysis in 9.1 can be applied here.

- simple autonomous system： $\dot{x} = \mathrm{A}x$ $\quad \mathrm{A}: 2\times 2$

Ex：

Consider an oscillating pendulum, by the conservation of angular momentum, the equation of the movement is written as



$$\ddot{\theta}+\frac{c}{mL}\dot{\theta}+\frac{g}{L}\sin\theta=0 \quad \text{or} \quad \ddot{\theta}+r\dot{\theta}+\omega^2\sin\theta=0.$$

To transform this second order equation into first order linear system, we let $x=\theta$, $y=\dot{\theta}$, then the system becomes

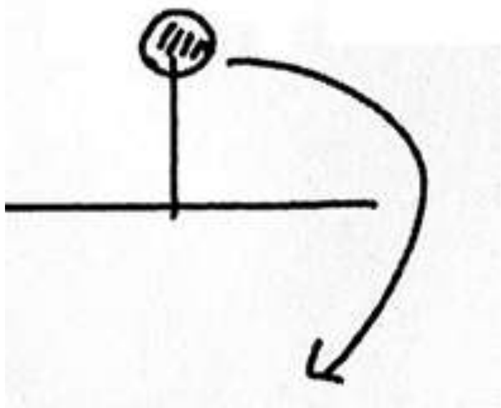
$$\begin{cases}\dot{x}=y \\ \dot{y}=\ddot{\theta}=-ry-\omega^2\sin x\end{cases} \qquad \therefore \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & -r\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}-\begin{pmatrix}0\\ \omega^2\sin x\end{pmatrix}$$

This is a nonlinear system.

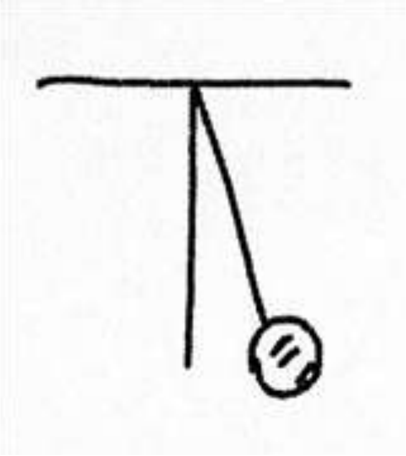
We can find the fixed point by setting

$$\begin{cases} y = 0 \\ -ry - \omega^2 \sin x = 0 \end{cases} \Rightarrow \begin{cases} x = \pm n\pi \rightarrow \theta \\ y = 0 \end{cases}$$

Can we determine the stability of these fixed points from intuition ?



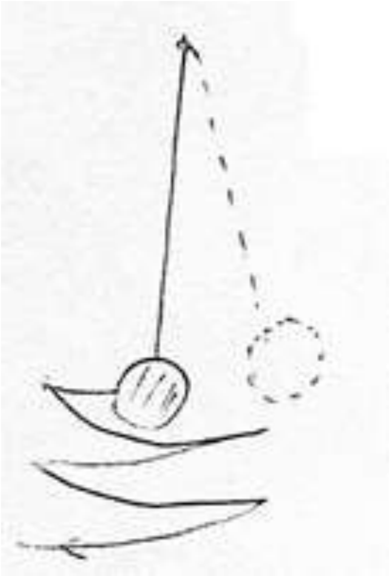
unstable



A.S.

These points correspond to two physical equilibrium positions, one with $\theta = 0$, the other one with $\theta = \pi$.

- Our intuition suggests that the first is stable and the second is unstable.
- If we don't consider the damping force (c or r is zero) and if the mass in displaced slightly from its lower equilibrium position, then it will oscillate indefinitely with constant amplitude about the equilibrium position.
- This type of motion is stable, but not asymptotically stable.



(without air resistance)

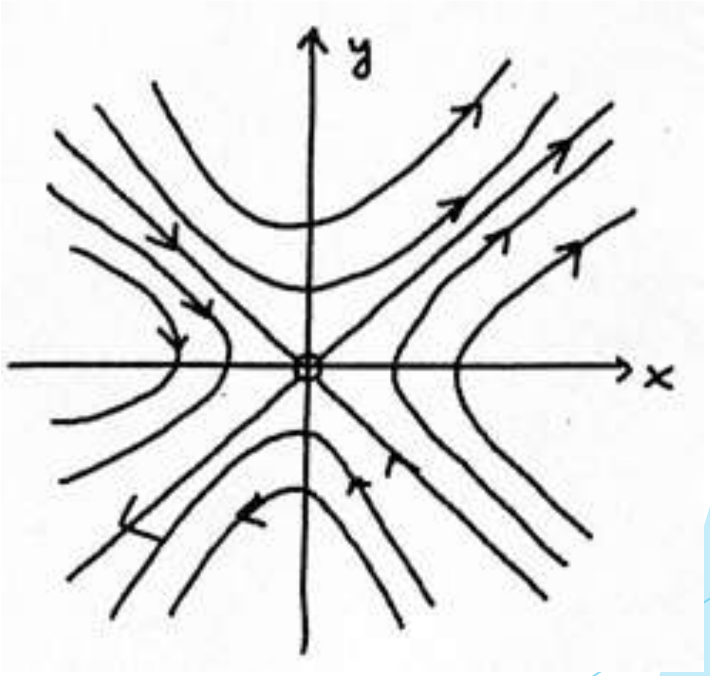
Ex 1 :
$$\begin{cases} \dot{x} = y & \times \quad x \\ \dot{y} = x & \times \quad y \end{cases}$$

(0 , (is the only critical point.

$$\Rightarrow \quad \dot{x}x - \dot{y}y = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}x^2 - \frac{1}{2}y^2\right) = 0$$

$$\Rightarrow \quad x^2 - y^2 = c$$



Ex 2： $\begin{cases} \dot{x} = 4 - 2y \\ \dot{y} = 12 - 3x^2 \end{cases}$

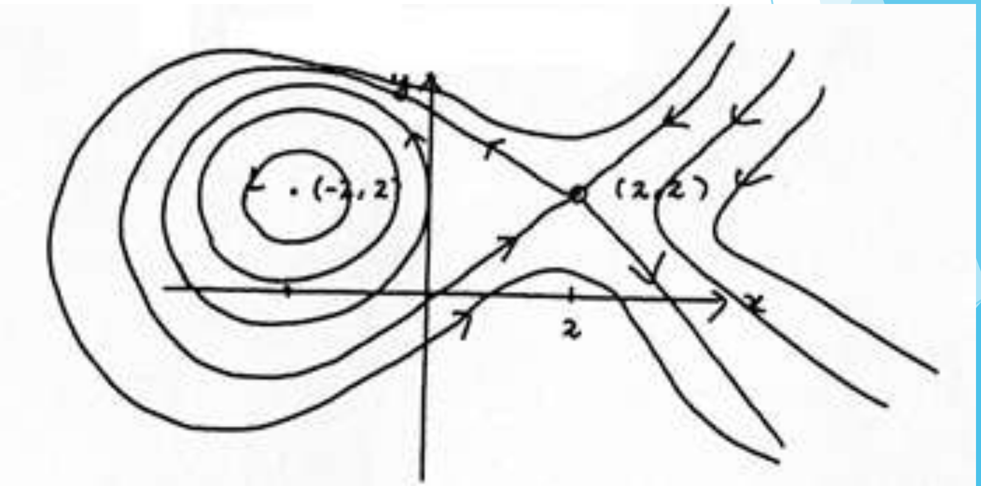
critical points： $(-2, 2)$, $(2, 2)$

$$\frac{dx}{dy} = \frac{4 - 2y}{12 - 3x^2}$$

$$\Rightarrow \left(12 - 3x^2\right) dx = \left(4 - 2y\right) dy$$

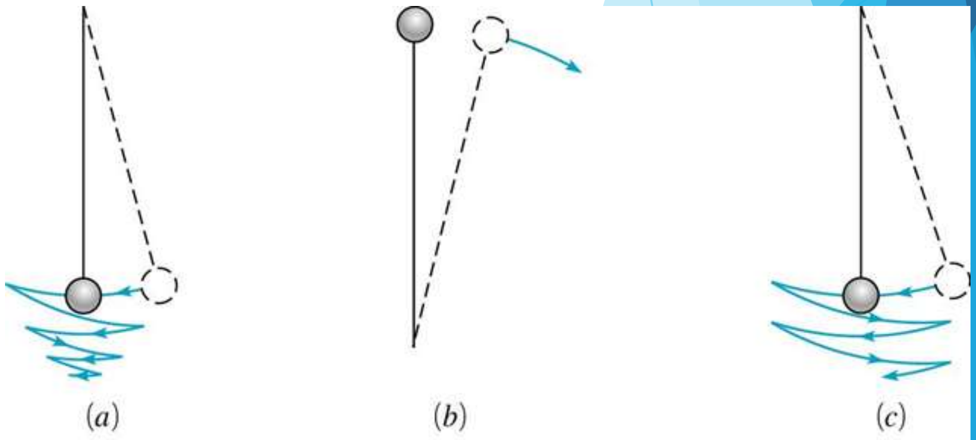
$$\Rightarrow 12x - x^3 + c = 4y - y^2$$

$$\therefore 4y - y^2 - 12x + x^3 = c$$



Stability of Critical Points: Damped Case

- If a mass is slightly displaced from a lower equilibrium position, it will oscillate with decreasing amplitude, and slowly approach an equilibrium position as the damping force dissipates initial energy. This type of motion illustrates asymptotic stability.
- If a mass is slightly displaced from an upper equilibrium position, it will rapidly fall, and then approach a lower equilibrium position. This type of motion illustrates instability
- See figures (a) and (b)

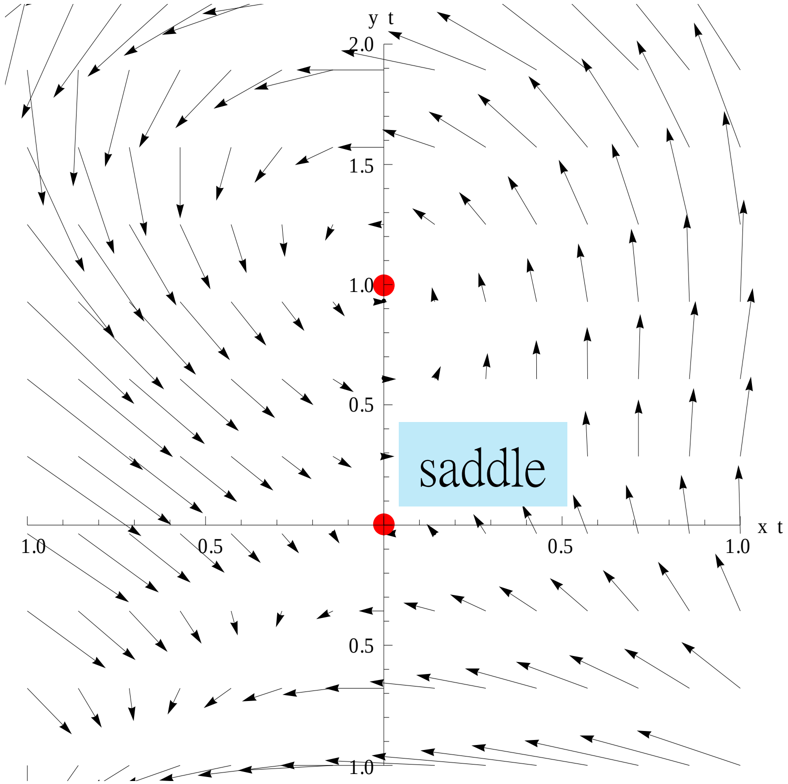


Example 1: Critical Points of a Nonlinear System (2 of 2)

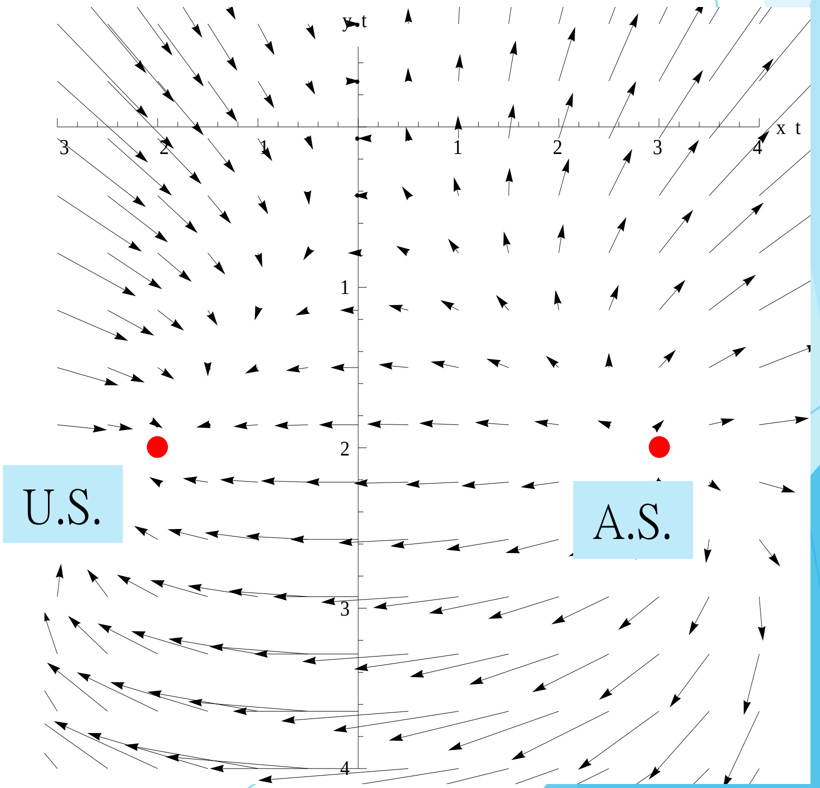
$\dot{x} = -(x-y)(1-x-y) \qquad \dot{y} = x(2+y)$ Four critical points： $(0,0)$, $(0,1)$, $(-2,-2)$, $(3,-2)$

- We can draw direction fields in the neighborhoods of these critical points to get a sense of the nature of these points.

Look at the points (0, 0) and (0, 1)

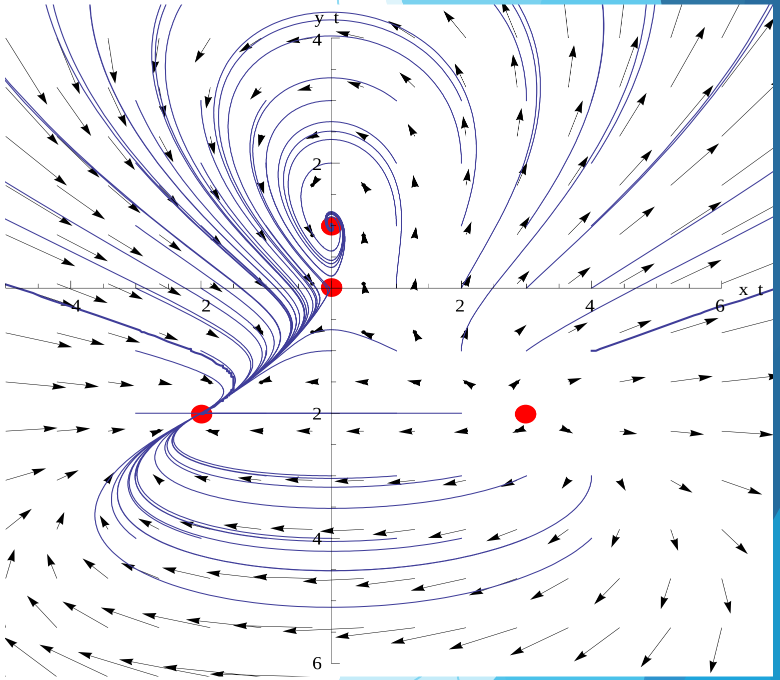


Look at the points (-2, -2) and (3, -2)



Example 2: Addition of Phase Portrait

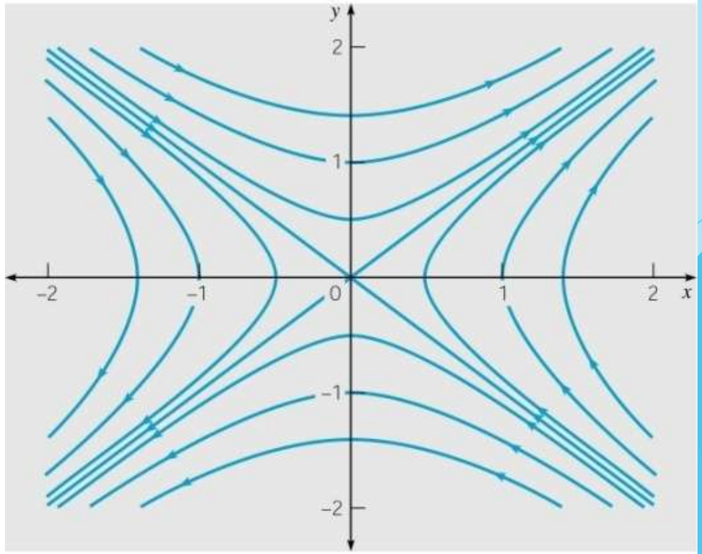
- See how the plot of trajectory solutions for the system in Example 1 fits in with the direction field plotted.
- You should notice there appears to be a saddle point at (0, 0) and an unstable equilibrium at (3, -2).
- The basin of attraction for the spiral node at (0,1) can be viewed and the rest of the plane is the basin of attraction for the asymptotically stable node at (-2,2).
- There are two separatrices both passing through the origin.
- Refer to Figure 9.2.6 in the text.



$$dx/dt = -(x-y)(1-x-y), \;\; dy/dt = x(2+y)$$

Example 3

- Consider the system $dx/dt = y, \;\; dy/dt = x$
- It follows that $dy/dx = x/y \;\; \Leftrightarrow \;\; ydy = x\,dx$
- The solution of this separable equation is $H(x, y) = y^2 - x^2 = c$
- Thus the trajectories are hyperbolas, as shown below.
- The direction of motion can by inferred from the signs of *dx*/*dt* and *dy*/*dt* in the four quadrants.



Example 4: Phase Portrait (2 of 2)

- We have $H(x,y) = 4y - y^2 - 12x + x^3 = c$
- A graph of some level curves of H are given below.
- Note that (-2, 2) is a center and (2, 2) is a saddle point.
- Also, one trajectory leaves the saddle point (at t = -∞), loops around the center, and returns to the saddle point (at t = ∞).