CHAPTER 2

Valuation

2.1. Contingent claim

Assumption. The law of one price holds.

Definition 2.1. A contingent claim is a random variable $Z : \mathbb{R}^S \longrightarrow [0, \infty)$. Throughout this chapter we may regard Z as a vector in $\mathbb{R}^S$ with $Z \geq 0$.

Remark 2.2. If the market is complete, i.e., $\mathcal{M} = \text{span}\{X_1, X_2, ..., X_J\} = \mathbb{R}^S$, then for $Z \in \mathbb{R}^S$, there exist $a_1, a_2, ..., a_J \in \mathbb{R}$ such that

$$Z = \sum_{j=1}^{J} a_j X_j,$$

then

$$q(Z) = q\left(\sum_{j=1}^{J} a_j X_j\right) = \sum_{j=1}^{J} a_j q(X_j) = \sum_{j=1}^{J} a_j P_j.$$

Note that the value of $q(Z)$ is unique, since q is a function.

Question. How about the case of incomplete market? There is a big problem: q is defined on $\mathcal{M}$. For general $Z \in \mathbb{R}^S$, how can we define it?

Definition 2.3. Define $q_u,\ q_l : \mathbb{R}^S \to \mathbb{R}$ by

$$q_u(Z) := \min_h \{h^T P : h^T X \geq Z\} \qquad \text{(upper bound)},$$

$$q_l(Z) := \max_h \{h^T P : h^T X \leq Z\} \qquad \text{(lower bound)}.$$

(Why we call them upper and lower bound, will be explained later.)

Remark 2.4. (1) $q_u(Z)$ is the lowest price of a portfolio, the payoff which dominates the contingent claim Z.

(2) $q_l(Z)$ is the highest price of a portfolio, the payoff which is dominated the contingent claim Z.

Proposition 2.5. *No strong arbitrage* $\implies$ $q_u(Z) = q(Z) = q_l(Z)$*, for all* $Z \in \mathcal{M}$*.*

Proof. Since $Z \in \mathcal{M}$, there exists a portfolio k such that $Z = k^T X$. Hence,

$$k^T P \geq \min\{h^T P : h^T X \geq Z\}.$$

By the definition of q_u and q_l, $q_u(Z) \leq q(Z)$. Similarly, we can get

$$q_u(Z) \leq q(Z) \leq q_l(Z) \qquad \text{for all } Z \in \mathcal{M}.$$

Suppose $q_u(Z) < q(Z)$ for some $Z \in \mathcal{M}$, then there exists a portfolio l such that

$$l^T X \geq Z \qquad \text{and} \qquad l^T P < q(Z).$$

Let h be a portfolio such that $h^T X = Z$ and $h^T P = q(Z)$. Then

$$\begin{aligned} (l-h)^T X &= l^T X - h^T X \geq Z - Z = 0, \\ (l-h)^T P &= l^T P - h^T P < q(Z) - q(Z) = 0. \end{aligned}$$

This means that $l - h$ is a strong arbitrage, which leads to a contradiction.

Thus, $q_u(Z) = q(Z)$ for all $Z \in \mathcal{M}$.

Similarly for proving $q_l(Z) = q(Z)$ for all $Z \in \mathcal{M}$. $\square$

Example 2.6. (1) Two states and one security with $P_1 = 1,\ X_1 = (1,2)$.

Consider the contingent claim $Z = (1,1)$ ($\Longrightarrow Z \notin \mathcal{M}$), then

$$q_u(Z) = \min\{h^T P_1 : h^T X \geq Z\} = \min\{h : (h, 2h) \geq (1,1)\} = 1,$$

$$q_l(Z) = \max\{h^T P_1 : h^T X \leq Z\} = \max\{h : (h, 2h) \leq (1,1)\} = \frac{1}{2}.$$

(2) Suppose that there are two securities and 3 states:

$$\text{security 1:} \quad P_1 = \frac{1}{2}, \quad X_1 = (1,1,1),$$

$$\text{security 2:} \quad P_2 = 1, \quad X_2 = (1,2,4).$$

Find $q_u(Z_1)$ and $q_l(Z_1)$ for $Z_1 = (0,0,1)$.

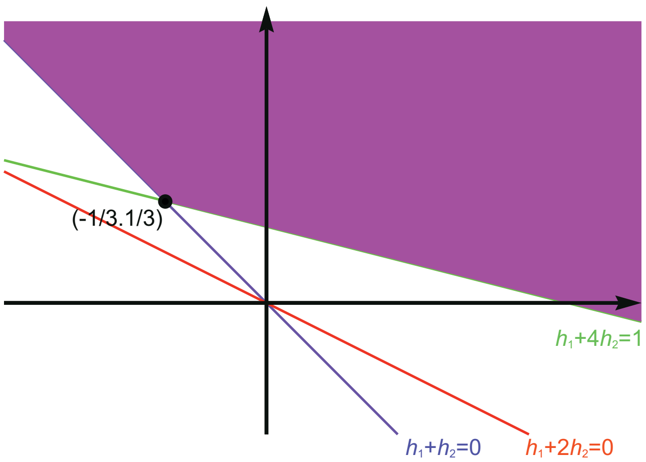

FIGURE 2.1

(a) By the definition, we have

$$\begin{aligned} q_u(0,0,1) &= \min\{h^T P : h^T X \geq Z_1\} \\ &= \min\left\{(h_1, h_2)\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} : (h_1, h_2)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \geq (0,0,1)\right\} \\ &= \min_{h_1.h_2}\left\{\frac{1}{2}h_1 + h_2 : (h_1 + h_2, h_1 + 2h_2, h_1 + 4h_2) \geq (0,0,1)\right\}, \end{aligned}$$

i.e., we aim to solve the optimization problem

$$\min_{h_1.h_2} \left(\frac{1}{2} h_1 + h_2 \right)$$

subject to the constraints

$$h_1 + h_2 \geq 0, \qquad h_1 + 2h_2 \geq 0, \qquad \text{and} \qquad h_1 + 4h_2 \geq 1.$$

By linear programming, we get the optimal point $= \left(-\frac{1}{3}, \frac{1}{3}\right)$ (See Figure 2.1). Thus,

$$\min_{h_1.h_2} \left(\frac{1}{2} h_1 + h_2 \right) = \frac{1}{2} \left(-\frac{1}{3} \right) + \frac{1}{3} = \frac{1}{6},$$

i.e., $q_u(0,0,1) = \frac{1}{6}$.

(b) Similarly as above, we have

$$q_l(0,0,1) = \max_{h_1.h_2} \left\{ \frac{1}{2} h_1 + h_2 : (h_1 + h_2, h_1 + 2h_2, h_1 + 4h_2) \leq (0,0,1) \right\},$$

i.e., we aim to solve the optimization problem

$$\max_{h_1.h_2} \left(\frac{1}{2} h_1 + h_2 \right)$$

subject to

$$h_1 + h_2 \leq 0, \qquad h_1 + 2h_2 \leq 0, \qquad h_1 + 4h_2 \leq 1.$$

Similar argument as above, we have $q_l(0,0,1) = 0$.

Moreover, we can compute some more examples:

(i) If $Z_2 = (1,0,0)$, then $q_u(1,0,0) = \frac{1}{3}$, $q_l(1,0,0) = 0$.

(ii) If $Z_3 = (1,0,1)$, then $q_u(1,0,1) = \frac{1}{2}$, $q_l(1,0,1) = 0$.

(3) Suppose that there are two securities and three states.

$$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}.$$

Consider a contingent claim $Z = (3, 4, 5)$. Since

$$Z = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix},$$

we get $Z \in \mathcal{M}$, but

$$\begin{aligned} q_u(Z) &= \min_h \left\{ h^T P : \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \geq \begin{pmatrix} 3 & 4 & 5 \end{pmatrix} \right\} \\ &= \min_{h_1, h_2} \{h_1 + h_2 : h_1 + 2h_2 \geq 3, h_1 + 3h_2 \geq 4, h_1 + 4h_2 \geq 5\} \\ &= -\infty. \end{aligned}$$

(See Figure 2.2 for its graph) and

$$\begin{aligned} q_l(Z) &= \max_h \left\{ h^T P : \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \leq \begin{pmatrix} 3 & 4 & 5 \end{pmatrix} \right\} \\ &= \infty. \end{aligned}$$

Remark 2.7. q_u and q_l are not linear. For example, as in Example 2.6(2), we have

$$q_u(0, 1, 0) = \frac{1}{2}, \qquad q_u(1, 0, 0) = \frac{1}{3}, \qquad q_u(1, 1, 0) = \frac{1}{2}.$$

This implies that

$$q_u(1, 1, 0) \neq q_u(1, 0, 0) + q_u(0, 1, 0),$$

i.e., q_u is not a linear functional.

Proposition 2.8. *No strong arbitrage* $\Longrightarrow$ $q_u(Z) \geq q_l(Z)$, *for all* $Z \in \mathbb{R}^S$.

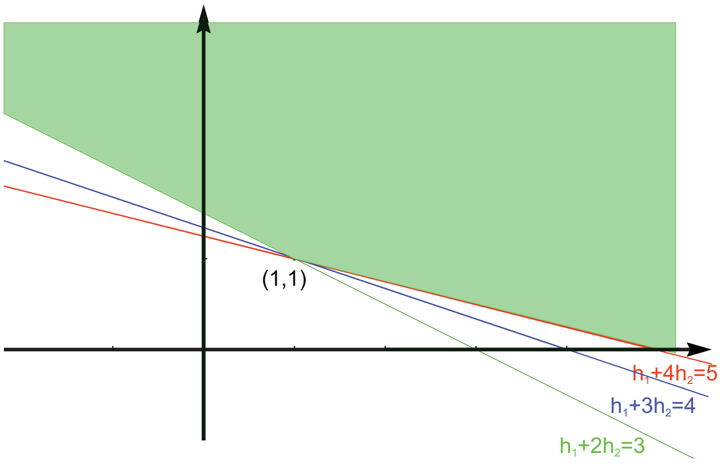

FIGURE 2.2

PROOF. Suppose that $q_u(Z) < q_l(Z)$ for some $Z \in \mathbb{R}^S$. By the definition of q_u and q_l, there exist portfolios m and n such that

$$m^T X \leq Z \leq n^T X \qquad \text{and} \qquad m^T P > n^T P.$$

This implies that

$$(n-m)^T X = n^T X - m^T X \geq 0,$$
$$(n-m)^T P = n^T P - m^T P < 0.$$

This means that $n - m$ is a strong arbitrage. $\square$

Proposition 2.9. *No arbitrage* $\implies$ $q_u(Z) > q_l(Z)$*, for all* $Z \notin \mathcal{M}$.

PROOF. We only have to prove that $q_u(Z) \neq q_l(Z)$, for all $Z \notin \mathcal{M}$. Suppose that $q_u(Z) = q_l(Z)$, for some $Z \notin \mathcal{M}$. Then there exist portfolios m and n such that

$$m^T X \leq Z \leq n^T X \qquad \text{and} \qquad m^T P = n^T P = q_u(Z).$$

This implies that

$$(n-m)^T X = n^T X - m^T X \geq 0,$$
$$(n-m)^T P = n^T P - m^T P = 0.$$

Since $Z \notin \mathcal{M}$,

$$m^T X < Z \leq n^T X \qquad \text{or} \qquad m^T X \leq Z < n^T X,$$

which means that $(n-m)^T X > 0$, i.e., $(n-m)$ is an arbitrage. □

Remark 2.10. Due to Propositions 2.5, 2.8, and 2.9 we see that if the market model is arbitrage-free,

(1) $q_u(Z) > q_l(Z)$, for all $Z \notin \mathcal{M}$.
(2) $q_u(Z) = q_l(Z) = q(Z)$, for all $Z \in \mathcal{M}$.

2.2. Valuation functional

Definition 2.11. A valuation functional is an extension of the payoff pricing functional on $\mathbb{R}^S$, i.e., the valuation functional is a linear functional $Q : \mathbb{R}^S \to \mathbb{R}$ and

$$Q(Z) = q(Z) \qquad \text{for all } Z \in \mathcal{M}. \tag{2.1}$$

Remark 2.12. (1) If the market is complete, the valuation functional and the payoff pricing functional are identical, i.e.,

$$Q(Z) = q(Z) = q_u(Z) = q_l(Z) \qquad \text{for all } Z \in \mathbb{R}^S.$$

(2) If the market is incomplete, the valuation functional is not unique. Both of the functional q_u and q_l satisfy (2.1), but they may not be valuation functionals, since they are not linear.

From this remark, we see why we construct this new functional instead of using directly the functionals q_u and q_l. The valuation functional must be linear. If the valuation functional is not linear, this causes an arbitrage in the model.

Example 2.13. Suppose that there are two states and a single security with payoff $X_1 = (1, 2)$ and $P_1 = 1$. The asset span

$$\mathcal{M} = \operatorname{span}\{(1, 2)\} = \{(\alpha, 2\alpha) : \alpha \in \mathbb{R}\}.$$

Thus, the payoff pricing function

$$q(\alpha, 2\alpha) = \alpha \cdot 1 = \alpha.$$

We aim to find $q_1, q_2 \in \mathbb{R}$ such that

$$Q(z_1, z_2) = q_1 z_1 + q_2 z_2 \qquad \text{and} \qquad Q(\alpha, 2\alpha) = q(\alpha, 2\alpha) = \alpha.$$

Plugging the later equation into the first one, we get

$$Q(\alpha, 2\alpha) = q_1\alpha + 2q_2\alpha = \alpha, \qquad \text{for all } \alpha.$$

This implies $q_1 + 2q_2 = 1$. Q is strictly positive, if $q_1, q_2 \geq 0$. Thus, each function $Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$Q(z_1, z_2) = q_1 z_1 + q_2 z_2,$$

with $q_1, q_2 > 0$, $q_1 + 2q_2 = 1$, is a strictly positive valuational functional, since Q is linear and

$$Q(\alpha, 2\alpha) = \alpha(q_1 + 2q_2) = \alpha.$$

Theorem 2.14 (Fundamental Theorem of Finance). *No arbitrage* $\iff$ *there exists a strictly positive valuation functional.*

PROOF. "$\Longleftarrow$" Suppose Q is a strictly positive valuation function, then q is a strictly positive payoff pricing functional, which implies that there is no arbitrage by Theorem 1.32.

"$\Longrightarrow$" Clearly, $\mathcal{M}$ is a subspace of $\mathbb{R}^S$. (Exercise)

Fixed $\hat{Z} \notin \mathcal{M}$, define

$$\mathcal{N} = \{Z + \lambda\hat{Z} : Z \in \mathcal{M}, \lambda \in \mathbb{R}\} = \mathcal{M} + \operatorname{span}(\{\hat{Z}\}).$$

Then $\mathcal{N}$ is also a subspace of $\mathbb{R}^S$.

If we can find a linear functional $Q : \mathcal{N} \longrightarrow \mathbb{R}$ to be strictly positive, then we can find a linear functional $Q : \mathbb{R}^S \longrightarrow \mathbb{R}$ also to be strictly positive.

Step 1: Definition of $Q : \mathcal{N} \longrightarrow \mathbb{R}$.

By the assumption of no arbitrage, we have $q_u(\hat{Z}) > q_l(\hat{Z})$ by Proposition 2.9. Choose

$$\pi \in \Big(q_l(\hat{Z}), q_u(\hat{Z})\Big)$$

and define $Q : \mathcal{N} \longrightarrow \mathbb{R}$ by

$$Q(Z + \lambda\hat{Z}) \equiv q(Z) + \lambda\pi. \tag{2.2}$$

Step 2: Claim: $Q : \mathcal{N} \longrightarrow \mathbb{R}$ is linear.

Let $y_1, y_2 \in \mathcal{N}$, then there exist $Z_1, Z_2 \in \mathcal{M}$, λ_1, $\lambda_2 \in \mathbb{R}$ such that

$$y_1 = Z_1 + \lambda_1\hat{Z} \qquad \text{and} \qquad y_2 = Z_2 + \lambda_2\hat{Z}.$$

Thus, for $c \in \mathbb{R}$, by (2.2) we have

$$\begin{aligned}
Q(cy_1 + y_2) &= Q\left(c\left(Z_1 + \lambda_1 \hat{Z}\right) + \left(Z_2 + \lambda_2 \hat{Z}\right)\right) \\
&= Q\left((cZ_1 + Z_2) + (c\lambda_1 + \lambda_2)\,\hat{Z}\right) \\
&= q\left((cZ_1 + Z_2)\right) + (c\lambda_1 + \lambda_2)\,\pi \\
&= (cq(Z_1) + q(Z_2)) + (c\lambda_1 + \lambda_2)\,\pi \\
&= c\left(q(Z_1) + \lambda_1 \hat{Z}\right) + \left(q(Z_2) + \lambda_2 \hat{Z}\right) \\
&= cQ\left(Z_1 + \lambda_1 \hat{Z}\right) + Q\left(Z_2 + \lambda_2 \hat{Z}\right) = cQ(y_1) + Q(y_2).
\end{aligned}$$

Step 3: Claim: $Q : \mathcal{N} \longrightarrow \mathbb{R}$ is strictly positive, i.e., for $Y \in \mathcal{N}$ with $Y > 0$, we aim to show that $Q(Y) > 0$.

For $Y \in \mathcal{N}$ and $Y > 0$, there exists $Z \in \mathcal{M}$, $\lambda \in \mathbb{R}$ such that $Y = Z + \lambda \hat{Z} > 0$.

(1) If $\lambda > 0$, then $\hat{Z} > -\dfrac{Z}{\lambda}$. Since q_l is increasing (Exercise),

$$q_l(\hat{Z}) \geq q_l\left(-\frac{Z}{\lambda}\right) = -\frac{1}{\lambda}\,q_l(Z) \underset{Z \in \mathcal{M}}{=} -\frac{1}{\lambda}\,q(Z).$$

Thus,

$$\pi > q_l(\hat{Z}) \geq -\frac{1}{\lambda}\,q(Z).$$

This implies that

$$Q(Y) == Q(Z + \lambda \hat{Z}) = q(Z) + \lambda\pi > 0.$$

(2) If $\lambda < 0$, similar argument for $\pi \leq q_u(\hat{Z})$. This implies that $Q(Y) > 0$.

(3) If $\lambda = 0$, $Y = Z$, thus, $Q(Y) = q(Z) > 0$, since q is strictly positive.

Thus, we can get that Q is strictly positive.

$\square$

Theorem 2.15 (Fundamental Theorem of Finance, Weak Form). *No strong arbitrage* $\Longleftrightarrow$ *there exists a positive valuation functional.*

Proof. Similar argument as in Theorem 2.14. Simply choose $\pi \in \left[q_l(\hat{Z}), q_u(\hat{Z})\right]$. $\square$

Example 2.16. (1) Suppose $P_1 = 1$, $\mathcal{M} = \text{span}\{(1,2)\}$ and $\hat{Z} = (1,1)$, then

$$\mathcal{N} = \{Z + \lambda\hat{Z} : Z \in \mathcal{M}, \lambda \in \mathbb{R}\} = \mathbb{R}^2.$$

By Example 2.6(1), we have $\frac{1}{2} < \pi < 1$. Choose $\pi = \frac{3}{4}$. Define $Q : \mathcal{N}(= \mathbb{R}^2) \longrightarrow \mathbb{R}$ by

$$Q(Z + \lambda\hat{Z}) = q(Z) + \frac{3}{4}\lambda.$$

For any $(y_1, y_2) \in \mathcal{N}$, we want to find some points $Z \in \mathcal{M}$ (i.e., $Z = (\alpha, 2\alpha)$ for some $\alpha \in \mathbb{R}$) and $\lambda \in \mathbb{R}$ such that

$$(y_1, y_2) = Z + \lambda\hat{Z} = (\alpha, 2\alpha) + \lambda(1,1) = (\alpha + \lambda, 2\alpha + \lambda).$$

Soving the system of equations

$$\begin{cases} \alpha + \lambda &= y_1, \\ 2\alpha + \lambda &= y_2, \end{cases}$$

we have $\alpha = y_2 - y_1$ and $\lambda = 2y_1 - y_2$. Due to $q(1,2) = 1$, $q(\alpha, 2\alpha) = \alpha$. This implies that

$$\begin{aligned} Q(y_1, y_2) &= Q((y_2 - y_1)(1,2) + (2y_1 - y_2)(1,1)) \\ &= (y_2 - y_1)q(1,2) + (2y_1 - y_2)\pi \\ &= (y_2 - y_1) \cdot 1 + (2y_1 - y_2) \cdot \frac{3}{4} = \frac{1}{2}y_1 + \frac{1}{4}y_2. \end{aligned}$$

Thus, this model is arbitrage-free.

(2) As in Remark 1.36, consider a model with two securities and three states

$$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Then its asset span is given by

$$\mathcal{M} = \{(x, y, -x+2y) : x, y \in \mathbb{R}\}$$

and the corresponding pricing valuation functional $q : \mathcal{M} \longrightarrow \mathbb{R}$ is of the form

$$q((x, y, -x+2y)) = \frac{1}{2}y = \left(0, \frac{1}{2}\right) \cdot (x, y),$$

i.e., $a = (0, 1/2) \geq 0$. Consider some verctor $\hat{Z} \in \mathbb{R}^3 \setminus \mathcal{M}$, e.g., we may choose $\hat{Z} = (0, 0, 1)$. Then

$$\begin{aligned} q_u(\hat{Z}) &= \min_h \left\{ P \cdot h : \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \geq \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \right\} \\ &= \min_{h_1 . h_2} \{h_1 + h_2 : 2h_1 + h_2 \geq 0, 2h_1 + 2h_2 \geq 0, 2h_1 + 3h_2 \geq 1\} = \frac{1}{4}. \end{aligned}$$

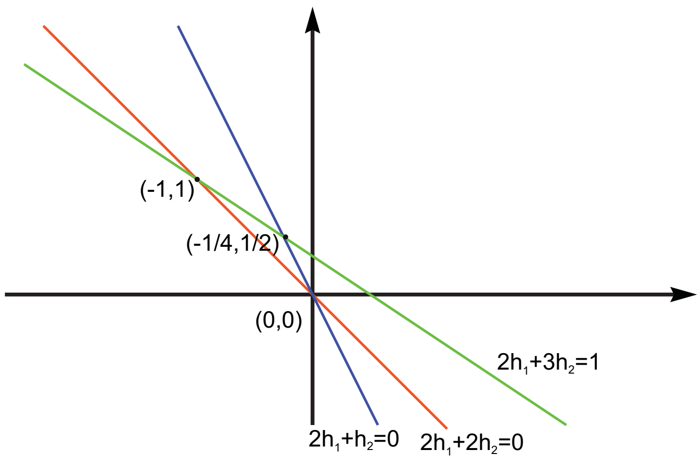

Figure 2.3

Similarly, we have

$$q_l(1,1) = \min_{q_1,q_2 \geq 0} \{q_1 + q_2 : q_1 + 2q_2 = 1\} = \frac{1}{2}.$$

Similarly, we have $q_l(\hat{Z}) = 0$. Therefore, $0 < \pi < \frac{1}{4}$, we may choose $\pi = \frac{1}{8}$. Thus,

$$\begin{aligned} Q(x,y,z) &= Q\left((x,y,-x+2y) + (0,0,z+x-2y)\right) \\ &= q((x,y,-x+2y)) + (z+x-2y)Q(0,0,1) = \frac{1}{2}\,y + \frac{1}{8}\,(z+x-2y) \\ &= \frac{1}{8}\,x + \frac{1}{4}\,y + \frac{1}{8}\,z = \left(\frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right) \cdot (x,y,z), \end{aligned}$$

which is strictly positive. Hence, there is no arbitrage in the model.

(3) As in Example 1.37, its payoff pricing functional q is given by

$$q\left(\left(x, y, -\frac{1}{3}\,x - \frac{1}{6}\,y\right)\right) = -x + 4y.$$

Consider a contingent claim $\hat{Z} = (0,0,1)$, then

$$\begin{aligned} q_u(\hat{Z}) &= \min_h \left\{ h^T P : \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 2 & 2 & -1 \end{pmatrix} \geq \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \right\} \\ &= \min_{h_1.h_2} \{9h_1 + 6h_2 : -h_1 + 2h_2 \geq 0, 2h_1 + 2h_2 \geq 0, h_2 \leq 0\} = 12 \end{aligned}$$

and

$$\begin{aligned} q_l(\hat{Z}) &= \max_h \left\{ h^T P : \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 2 & 2 & -1 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \right\} \\ &= \max_{h_1.h_2} \{9h_1 + 6h_2 : -h_1 + 2h_2 \leq 0, 2h_1 + 2h_2 \leq 0, h_2 \geq 0\} = 3 \end{aligned}$$

(see Figure 2.4). thus, we may choose $\pi \in (3,12)$, e.g., $\pi = 6$.

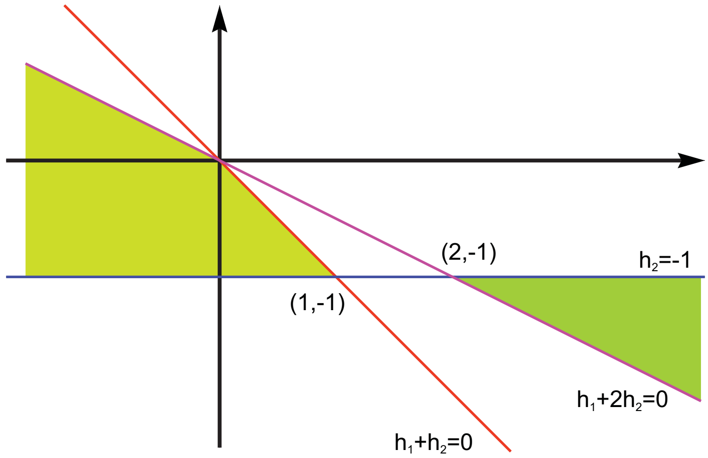

FIGURE 2.4

Then its valuation functional is of the form

$$\begin{aligned} Q(x,y,z) &= Q\left(\left(x, y, -\frac{1}{3}x - \frac{1}{6}y\right) + \left(z + \frac{x}{3} + \frac{y}{6}\right)(0,0,1)\right) \\ &= -x + 4y + 6\left(z + \frac{x}{3} + \frac{y}{6}\right) = x + 5y + 6z = (1,5,6)\cdot(x,y,z). \end{aligned}$$

This is an alternative method to check that this model is arbitrage-free.

An alternative (and much fast!) method to find an arbitrage free valuation functional is to combine Example 2.13 and Theorem 2.14. If we want to prove that a model is no arbitrage, we need to find a strictly positive functional Q, i.e, find a strictly positive vector $a \ll 0$ such that $Q(Z) = a \cdot Z$, such that $Q(Z) = q(Z)$ for all $Z \in \mathcal{M}$.

Example 2.17. (1) As in Example 2.13 and Example 2.16 (1), we know the pay-off pricing functional is given by

$$q(\alpha, 2\alpha) = \alpha.$$

We want to know if the model is arbitrage-free. That is, we have to find a strictly positive valuation functional. By Example 2.13, the valuation functional is of the form

$$Q(z_1, z_2) = q_1 z_1 + q_2 z_2,$$

with $q_1 + 2q_2 = 1$. Thus, the functional Q is strictly positive if and only if $q_1 + 2q_2 = 1$ with $q_1, q_2 > 0$.

(a) We may choose $q_1 = q_2 = \frac{1}{3}$. Hence,

$$Q(z_1, z_2) = \frac{1}{3}\, z_1 + \frac{1}{3}\, z_2$$

is a strictly positive valuation functional in this model.

(b) If we choose $q_1 = \frac{1}{2}$ and $q_2 = \frac{1}{4}$, then the corresponding valuation functional is of the form

$$Q(z_1, z_2) = \frac{1}{2}\, z_1 + \frac{1}{4}\, z_2,$$

the same as ginve in Example 2.16 (1).

(2) As in Example 1.37 and Example 2.16 (3), its payoff pricing functional q is given by

$$q\left(\left(x, y, -\frac{1}{3}\, x - \frac{1}{6}\, y\right)\right) = -x + 4y.$$

If

$$Q(x, y, z) = a_1 x + a_2 y + a_3 z$$

is a valuation functional, then

$$\begin{aligned} Q\left(\left(x, y, -\frac{1}{3}\, x - \frac{1}{6}\, y\right)\right) &= a_1 x + a_2 y + a_3 \left(-\frac{1}{3}\, x - \frac{1}{6}\, y\right) \\ &= \left(a_1 - \frac{1}{3}\, a_3\right) x + \left(a_2 - \frac{1}{6}\, a_3\right) y \\ &= q\left(\left(x, y, -\frac{1}{3}\, x - \frac{1}{6}\, y\right)\right) = -x + 4y \end{aligned}$$

Thus, we want to find a strictly positive soultion to

$$a_1 - \frac{1}{3}a_3 = -1 \qquad \text{and} \qquad a_2 - \frac{1}{6}a_3 = 4.$$

We may find $(a_1, a_2, a_3) = (1, 5, 6)$ or $(3, 6, 12)$ are such solutions. This means, the functional Q given by

$$x + 5y + 6z \qquad \text{or} \qquad 3x + 6y + 12z$$

is a strictly positive valuation functional.

Corollary 2.18. (1) *If there is no arbitrage, the market is complete.* $\iff$ *there exists a unique strictly positive valuation functional.*

(2) *If there is no strong arbitrage, the market is complete.* $\iff$ *there exists a unique positive valuation functional.*

Proof. Exercise. □

Remark 2.19. (1) If the model is arbitrage free, then $Q(Z) = q(Z) = q_l(Z) = q_u(Z)$ for all $Z \in \mathcal{M}$ and $Q(Z) \in (q_l(Z), q_u(Z))$ for all $Z \in \mathbb{R}^S \setminus \mathcal{M}$.

(2) If the model is strong arbitrage free, then $Q(Z) \in [q_l(Z), q_u(Z)]$ for all $Z \in \mathbb{R}^S$.

Example 2.20. In general, the valuation functional is not unique. For example, as in Example 2.16(2), if we choose $\pi = \dfrac{1}{6}$, then the corresponding valuation functional

$$Q(x, y, z) = \frac{1}{2}y + \frac{1}{6}(z + x - 2y) = \frac{1}{6}x + \frac{1}{6}y + \frac{1}{6}z.$$

2.3. Exercise

(1) Find $q_u(\hat{Z})$ and $q_l(\hat{Z})$ and one valuation functional $Q(Z)$ such that the model is arbitrage-free in the following cases.

(a) Two states and one security with $P_1 = 1$, $X_1 = (2, 2)$. The contingent claim $\hat{Z} = (2, 1)$.

(b) Two states and one security with $P_1 = 1$, $X_1 = (1, 3)$. The contingent claim $\hat{Z} = (2, 1)$.

(c) Three states and one security with $P_1 = 1$, $X_1 = (1, 2, 3)$. The contingent claim $\hat{Z} = (2, 1, 1)$.

(d) Three states and two securities with $P_1 = 1$, $X_1 = (2, 2, 2)$, $P_2 = 1$, $X_2 = (3, 1, 4)$. The contingent claim $\hat{Z} = (1, 2, 3)$.

(2) Suppose that there are three states and two securities with

$$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \qquad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

(a) Consider $Z_1 = (4, 2, 1)$, find $q_u(Z_1)$ and $q_l(Z_1)$.

(b) Find a strictly positive valuation functional.

(c) Consider $Z_2 = (1, 3, 3)$, find $q_u(Z_2)$ and $q_l(Z_2)$.

(d) Discuss the relation of the valuation function constructed in (b) and the functionals $q_u(Z_2)$ and $q_l(Z_2)$ in (c).

(3) Prove that the asset span $\mathcal{M}$ is a subspace of $\mathbb{R}^S$.

(4) Prove that the functions $q_u(Z)$ and $q_l(Z)$ are increasing in Z, i.e., for $Z_1 \geq Z_2$,

$$q_u(Z_1) \geq q_u(Z_2) \qquad \text{and} \qquad q_l(Z_1) \geq q_l(Z_2).$$

(5) Prove that

(a) if there is no arbitrage, the market is complete. $\iff$ there exists a unique strictly positive valuation functional.

(b) if there is no strong arbitrage, the market is complete. $\iff$ there exists a unique positive valuation functional.