CHAPTER 5

Expected Utility

5.1. Utility functions

In economics, we ofte describe prefrence relations by means of a utility function. A utility function $U(x)$ assigns a numerical value to each element in $\mathcal{X}$, ranking the elements in $\mathcal{X}$ in accordance with the individual's preference, i.e.,

$$x \succ y \qquad \Longleftrightarrow \qquad U(x) \geq U(y).$$

We call the function U a utility function.

Definition 5.1. (1) U is said to exhibit risk neutral if and only if U is affine (see Figure 5.1).

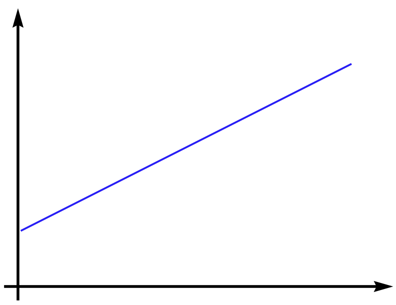

FIGURE 5.1. risk neutral utility function

(2) U is said to exhibit risk aversion if and only if U is strictly concave (see Figure 5.2).

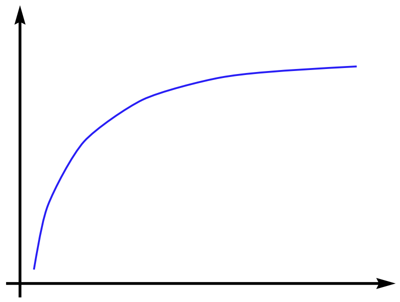

FIGURE 5.2. risk averse utility function

(3) U is said to exhibit risk seeking if and only if U is strictly convex (see Figure 5.3).

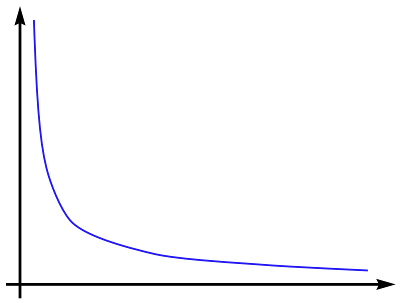

FIGURE 5.3. risk seeking utility function

As usual we assume that U is concave. The reason see the following figure (Figure 5.4).

5.2. Expected utility

Definition 5.2. Let $U : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a utility function,

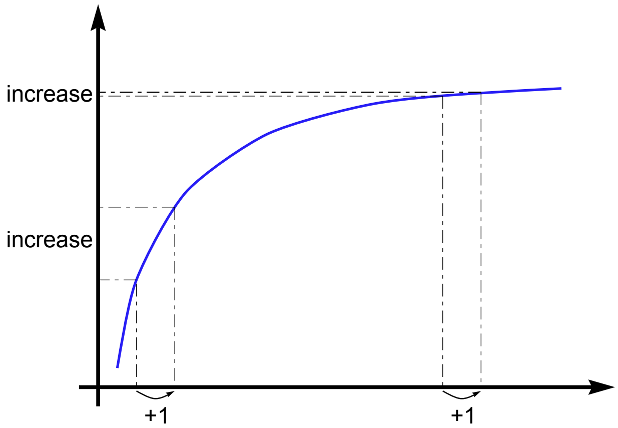

FIGURE 5.4. concave utility function

(1) U has a state-dependent expected utility representation if there exist functions $u_s : \mathbb{R} \longrightarrow \mathbb{R}$ for all $1 \leq s \leq N$ and a probability measure π on S such that

$$U(C_1, C_2, \cdots, C_N) \geq U(C_1', C_2', \cdots, C_N') \iff \sum_{s=1}^{N} \pi_s u_s(C_s) \geq \sum_{s=1}^{N} \pi_s u_s(C_s'),$$

where $C_i, C_i' \in S$ for all $1 \leq i \leq N$.

(2) U has a state-independent expected utility representation if there exist a function $u : \mathbb{R} \longrightarrow \mathbb{R}$ and a probability measure π on S such that

$$U(C_1, C_2, \cdots, C_N) \geq U(C_1', C_2', \cdots, C_N') \iff \sum_{s=1}^{N} \pi_s u(C_s) \geq \sum_{s=1}^{N} \pi_s u(C_s'), \qquad (5.1)$$

where $C_i, C_i' \in S$ for all $1 \leq i \leq N$.

Definition 5.3. The function u defined by (5.1) is called the von Neumann-Morgenstern utility function.

Remark 5.4. The definitions in Definition 5.2 and Definition 5.3 can be generalized to the case where the domain of U is infinite dimensional. Then Equation (5.1) is similar to the von-Neumann-Morgenstern representation introduced in Chapter 4.

Notation 5.5. When we consider π as a probability measure, the state space S can be regarded as a probability space. Thus, in the case without confusion, we may write $\sum_{s=1}^{S} \pi_s u(C_s)$ as $E_\pi[u(C)]$ or $E[u(C)]$.

Definition 5.6. (1) An agent with von Neumann-Morgenstern utility function $u : \mathbb{R} \longrightarrow \mathbb{R}$ is risk-averse if

$$E[u(C)] \leq u(E[C])$$

for every consumption plan C.

(2) An agent is risk-neutral if

$$E[u(C)] = u(E[C])$$

for every consumption plan C.

(3) An agent is risk-seeking if

$$E[u(C)] \geq u(E[C])$$

for every consumption plan C.

Theorem 5.7. (1) *An agent is risk-averse* $\Longleftrightarrow$ *His von Neumann-Morgenstern utility function u is concave.*

(2) *An agent is risk-neutral* $\Longleftrightarrow$ *His von Neumann-Morgenstern utility function u is affine.*

(3) *An agent is risk-seeking* $\Longleftrightarrow$ *His von Neumann-Morgenstern utility function u is convex.*

Proof. (1) "$\Longleftarrow$" If u is concave, then

$$E[u(C)] \leq u(E[C])$$

due to Jensen's inequality. Hence, the agent is risk averse.

"$\Longrightarrow$" Suppose that the agent is risk averse, but u is not concave. Then there exist y_1, y_2 and $\lambda^* \in (0,1)$ such that

$$u(\lambda y_1 + (1-\lambda)y_2) < \lambda^* u(y_1) + (1-\lambda^*)u(y_2).$$

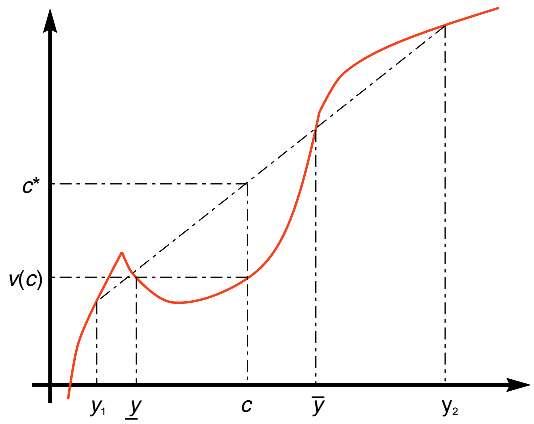

FIGURE 5.5. $c = \lambda^* y_1 + (1-\lambda^*)y_2$, $c^* = \lambda^* u(y_1) + (1-\lambda^*)u(y_2)$

Let

$$A = \{\lambda \in [0,\lambda^*] : u(\lambda y_1 + (1-\lambda)y_2) = \lambda u(y_1) + (1-\lambda)u(y_2)\},$$

then $A \neq \emptyset$ $(\lambda = 0 \in A)$ and A is closed (since v is continuous). Thus,

$$\underline{\lambda} = \sup A \quad \text{exists} \qquad \text{and} \qquad \underline{\lambda} < \lambda^*.$$

Similarly, there exists

$$\bar{\lambda} = \inf\{\lambda \in [\lambda^*,1] : u(\lambda y_1 + (1-\lambda)y_2) = \lambda u(y_1) + (1-\lambda)u(y_2)\},$$

and $\lambda^* < \bar{\lambda}$. This implies that

$$u(\lambda y_1 + (1-\lambda)y_2) < \lambda u(y_1) + (1-\lambda)u(y_2) \qquad \text{for all } \underline{\lambda} < \lambda < \bar{\lambda}.$$

Let

$$\underline{y} = \underline{\lambda} y_1 + (1-\underline{\lambda}) y_2 \qquad \text{and} \qquad \bar{y} = \bar{\lambda} y_1 + (1-\bar{\lambda}) y_2.$$

Then

$$\begin{aligned}
u(\gamma\underline{y} + (1-\gamma)\bar{y}) &= u(\gamma(\underline{\lambda} y_1 + (1-\underline{\lambda})y_2) + (1-\gamma)(\bar{\lambda} y_1 + (1-\bar{\lambda})y_2)) \\
&= u\left((\gamma\underline{\lambda} + (1-\gamma)\bar{\lambda})y_1 + (\gamma(1-\underline{\lambda}) + (1-\gamma)(1-\bar{\lambda}))y_2\right) \\
&= u((\underbrace{\gamma\underline{\lambda} + (1-\gamma)\bar{\lambda}}_{\in(\underline{\lambda},\bar{\lambda})})y_1 + (1-(\gamma\underline{\lambda} + (1-\gamma)\bar{\lambda})y_2) \\
&< (\gamma\underline{\lambda} + (1-\gamma)\bar{\lambda})u(y_1) + (1-(\gamma\underline{\lambda} + (1-\gamma)\bar{\lambda})u(y_2) \\
&= \gamma(\underline{\lambda} u(y_1) + (1-\underline{\lambda})u(y_2)) + (1-\gamma)(\bar{\lambda} u(y_1) + (1-\bar{\lambda})u(y_2)) \\
&= \gamma u(\underline{\lambda} y_1 + (1-\underline{\lambda})y_2) + (1-\gamma)u(\bar{\lambda} y_1 + (1-\bar{\lambda})y_2) \\
&= \gamma u(\underline{y}) + (1-\gamma)u(\bar{y}).
\end{aligned}$$

Let C take value $\underline{y}$ in some (not all) states and $\bar{y}$ in the remaining states. This implies that

$$u(E[C]) < E[u(C)],$$

which contradicts to the assumption of the risk aversion.

(2), (3): Exercise. □

Definition 5.8. If the von-Neumann-Morgenstern utility function u is strictly concave, strictly increasing, and continuous on S, the von-Neumann-Morgenstern representation is called an expected utility representation.

5.3. Expected utility and preference order

Definition 5.9. (1) A prefrence relation $\succ$ on $\mathcal{M}$ is called monotone if

$$x > y \quad \text{implies} \quad \delta_x \succ \delta_y.$$

(2) The preference relation $\succ$ is called risk averse if for $\mu \in \mathcal{M}$

$$\delta_{m(\mu)} \succ \mu \qquad \text{unless } \mu = m(\mu),$$

where $m(\mu) = \int x\mu(dx)$.

Proposition 5.10. *Suppose the preference relation $\succ$ has a von Neumann-Morgenstern representation*

$$U(\mu) = \int u(x)\,\mu(dx).$$

Then

(1) $\succ$ *is monotone if and only if u is strictly increasing.*

(2) $\succ$ *is risk averse if and only if u is strictly concave.*

Proof. (1) Mnotonicity is equivalnet to

$$u(x) = U(\delta_x) > U(\delta_y) = u(y).$$

(2) **"$\Longrightarrow$":** If $\succ$ is risk averse, then

$$\delta_{\alpha x+(1-\alpha)y} \succ \alpha\delta_x + (1-\alpha)\delta_y,$$

holds for all distinct $x, y \in S$ and $\alpha \in (0,1)$. Hence,

$$u\left(\alpha x + (1-\alpha)y\right) > \alpha u(x) + (1-\alpha)u(y),$$

i.e., u is strictly concave.

“$\Longleftarrow$”: If u is strictly concave, then Jensen's ineuality implies risk aversion:

$$U\left(\delta_{m(\mu)}\right) = u\left(m(\mu)\right) \geq \int u(x)\,\mu(dx) = U(\mu)$$

with equality if and only if $\mu = \delta_{m(\mu)}$.

□

Example 5.11. (St. Petersburg Paradox) Consider the lottery

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \delta_{2^{n-1}}$$

which may be viewed as the payoff distribution of the following game. A fair coin is tossed until a head appear. If the head appears in the nth toss, the payoff will be 2^{n-1}. What is the fair price?

- Up to the early 18th century, the fair price = the expected value of $\mu = \infty$. But it is hard to find someone who is ready to pay even 20.
- G. Cramer: Using the utility function $u(x) = \sqrt{x}$, then

$$\int u(x)\,\mu(dx) = \frac{1}{2-\sqrt{2}}.$$

  The fair price should be $\left(\frac{1}{2-\sqrt{2}}\right)^2 \sim 2.91$
- D. Bernoulli: Using the utility function $u(x) = \log x$, then the fair price is $\exp(\log 2) = 2$.

We will explain in the next chapter why they use these values as the fair prices.

Remark 5.12. A von-Neumann-Morgenstern utility function should look like in Figure 5.6. In fact, Tversky and Kahneman [**11**] suggests that the function u should be of the form

$$u(x) = \begin{cases} (x-c)^{0.88}, & \text{if } x \geq c, \\ -2.25(c-x)^{0.88}, & \text{if } x < c, \end{cases}$$

where c is a given benchmark level, called the reference point. More details will be discussed in Chapter 9.

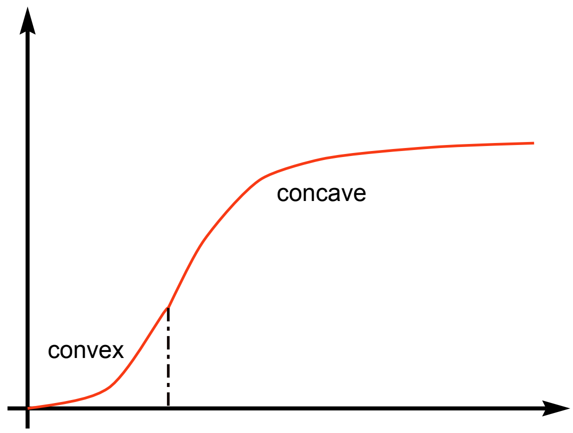

FIGURE 5.6. utility function

Example 5.13 (Ellsberg's paradox). Suppose that you have an urn containing 30 yellow balls and 60 other balls that are either red or blue. You don't know how many red or black balls there are, but there the total number of red balls plus the total number of blue balls equals 60. The balls are well mixed so that each individual ball is as likely to be drawn as any other. You are now given a choice between two gambles:

$$A \ : \ \begin{cases} \text{win \$2000} & \text{if you draw a yellow ball,} \\ \text{win \$0} & \text{otherwise,} \end{cases}$$

$$B \ : \ \begin{cases} \text{win \$2000} & \text{if you draw a blue ball,} \\ \text{win \$0} & \text{otherwise.} \end{cases}$$

Also you are given the choice between these two gambles (about a different draw from the same urn):

$$C \ : \ \begin{cases} \text{win } \$2000 & \text{if you draw a yellow or red ball,} \\ \text{win } \$0 & \text{otherwise,} \end{cases}$$

$$D \ : \ \begin{cases} \text{win } \$2000 & \text{if you draw a blue or red ball,} \\ \text{win } \$0 & \text{otherwise.} \end{cases}$$

For the choice between Gamble A and Gamble B, the most people would choose Gamble A ($A : B = 33 : 14$). However, for the choice Gamble C and Gamble D, the most people would choose Gamble D ($C : D = 5 : 39$). Mathematically, your estimated probabilities of each color ball can be represented as: p_Y, p_R, p_B. If you strictly prefer Gamble A to Gamble B, by expected utility theory, it is presumed this preference is reflected by the expected utilities of the two gambles: specifically, it must be the case that

$$p_Y u(2000) + (1 - p_Y)u(0) > p_B u(2000) + (1 - p_B)u(0).$$

Thus,

$$(p_Y - p_B)u(2000) > (p_Y - p_B)u(0). \tag{5.2}$$

If you also strictly prefer Gamble D to Gamble C, the following inequality is similarly abtained

$$p_R u(2000) + p_Y u(2000) + p_B u(0) > p_R u(2000) + p_B u(2000) + p_Y u(0).$$

This simplifies to

$$(p_Y - p_B)u(2000) < (p_Y - p_B)u(0).$$

This contradicts to (5.2). Thus, your preferences are inconsistent with the expected utility theory.

5.4. Maximal utility

Suppose that there are 2 securities and S states.

one bond: $P_1 = 1,\qquad X_1 = (1+r, 1+r, \cdots, 1+r) = (1+r)I$

one stock: $P_2 = P_S,\qquad X_2 = (D_1, D_2, \cdots, D_S) = D$

The initial endowment $= w$ and the trading portfolio $= (m, n)$, Thus, at time 0 we have the constraint $m \cdot P_1 + n \cdot P_2 = w$, i.e.,

$$m + nP_S = w.$$

The payoff at time 1 is given by

$$\begin{aligned} m(1+r)I + nD &= m(1+r, 1+r, ..., 1+r) + n(D_1, D_2, ..., D_S) \\ &= (m(1+r) + nD_1, m(1+r) + nD_2, ..., m(1+r) + nD_S) \end{aligned}$$

Thus, at state s the payoff at time 1 is of the form

$$m(1+r) + nD_s = (w - nP_s)(1+r) + nD_s.$$

Goal: $\max\limits_{m,n} E[u(m(1+r)I + nD)]$ subject to the constraint $m + nP_S = w$.

In other words, we consider

$$\max_n E[u(w(1+r) - (1+r)nP_S + nD)].$$

Due to the first order condition, we have

$$E[u'\left(w(1+r) - (1+r)nP_S + nD\right) \cdot (D - (1+r)P_S))] = 0.$$

This implies that

$$P_S = \frac{1}{1+r} \frac{E[u'(w(1+r) - nP_S(1+r) + nD) \cdot D]}{E[u'(w(1+r) - nP_S(1+r) + nD)]}.$$

We have n on the right hand side and n is changeable. Therefore, we have still some problems in this equation.

Proposition 5.14. *Assume that u is strictly concave (strictly risk averse)*

(1) *If* $P_S = \dfrac{1}{1+r}E[D]$, *then* $n = 0$.
(2) *If* $P_S < \dfrac{1}{1+r}E[D]$, *then* $n > 0$.
(3) *If* $P_S > \dfrac{1}{1+r}E[D]$, *then* $n < 0$ *(if short sale is allowed).*

Proof. Due to the first order condition,

$$0 = E\left[u' \cdot (D - (1+r)P_S)\right] = E\left[u' \cdot D\right] - (1+r)P_S E\left[u'\right].$$

Moreover,

$$\begin{aligned} E\left[u' \cdot D\right] &= E\left[u'\right]E\left[D\right] - E\left[u'E\left[D\right]\right] + E\left[u' \cdot D\right] \\ &= E\left[u'\right]E\left[D\right] + E\left[u'\left(D - E\left[D\right]\right)\right] \\ &= E\left[u'\right]E\left[D\right] + E\left[\left(u' - E\left[u'\right]\right)\left(D - E\left[D\right]\right)\right] \\ &= E\left[u'\right]E\left[D\right] + \mathrm{Cov}(u', D). \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= E\left[u' \cdot D\right] - (1+r)P_S E\left[u'\right] \\ &= E\left[u'\right]E\left[D\right] + \mathrm{Cov}(u', D) - (1+r)P_S E\left[u'\right] \\ &= \left(E\left[D\right] - (1+r)P_S\right)E\left[u'\right] + \mathrm{Cov}(u', D), \end{aligned}$$

where $\mathrm{Cov}(u', D) = \mathrm{Cov}\left(u'(w(1+r) - (1+r)nP_S + nD), D\right)$. Thus,

(1) $n = 0 \underset{\text{Exercise}}{\iff} \mathrm{Cov}(u', D) = 0 \iff P_S = \dfrac{1}{1+r} E[D].$

We know the precise stock price and the investor considers whether they should buy the stock. If they buy, there is risk.

(2) $n > 0 \underset{(*)}{\iff} \mathrm{Cov}(u', D) < 0 \iff P_S < \dfrac{1}{1+r} E[D].$

The stock is cheaper, the investor will buy and hold a little bit.

(3) $n < 0 \iff \mathrm{Cov}(u', D) > 0 \iff P_S > \dfrac{1}{1+r} E[D].$

Since the stock is too expensive, the investor will sell it.

$(*)$: Here use the fact that u is strictly concave. Hence $\mathrm{Cov}(u', D)$ is decreasing in n Moreover, if $n = 0$, $\mathrm{Cov}(u', D) = 0$, we can get the desired result. $\square$

The more general case about this model will be discussed in Chapter 8.

Example 5.15. Consider a market model with two securities and three states:

$$P_1 = 1, \quad X_1 = (2, 2, 2); \quad P_2 = 1, \quad X_2 = (1, 2, 4).$$

The probability is given by $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ and the initial endowment is 10. Hence,

(i) at time 0, $m + n = 10$;

(ii) at time 1, the payoff is $mX_1 + nX_2$.

Let u be a von Neumann-Morgenstern utility function. Thus, our goal is to maximize $E[u(mX_1 + nX_2)]$ under the constraint $m + n = 10$. In other words, we aim to solve the optimization problem

$$\max_n E[u(10X_1 + n(X_2 - X_1))].$$

(1) Suppose that $u(x) = x$ and $m, n \geq 0$ (This implies that $0 \leq m, n \leq 10$). Then

$$\begin{aligned} E[u(10X_1 + n(X_2 - X_1))] &= E[u((20-n, 20, 20+2n))] \\ &= \frac{1}{4}(20-n) + \frac{1}{2}\cdot 20 + \frac{1}{4}(20+2n) = 20 + \frac{1}{4}n \\ &\stackrel{!}{=} \max . \end{aligned}$$

Hence, the optimal trading portfolio is given by $(m, n) = (0, 10)$.

(2) Suppose that $u(x) = \ln x$. Then we want to find an optimal strategy such that

$$\begin{aligned} E[u(10X_1 + n(X_2 - X_1))] &= E[u((20-n, 20, 20+2n))] \\ &= \frac{1}{4}\ln(20-n) + \frac{1}{2}\ln 20 + \frac{1}{4}\ln(20+2n) \\ &= \frac{1}{2}\ln 20 + \frac{1}{4}\ln(20-n)(20+2n) \\ &\stackrel{!}{=} \max, \end{aligned}$$

which is equivalent to

$$\max \stackrel{!}{=} (20-n)(20+2n) = -2n^2 + 20n - 400.$$

The optimal solution to the above problem is given by $n = 5$. Thus, the optimal trading portfolio is given by $(m, n) = (5, 5)$.

5.5. Exercise

(1) Find utility functions for the following sets.

(a) $A = \{$all human being$\}$, living longer is better.

(b) $B = \{$all human being$\}$, more hair is preferred.

(c) $C = \{x°C : x \in [-273.12, 10000]\}$, near $28°C$ is preferred.

(d) $D = \{$all human being$\}$, rich is better.

(2) Suppose there are one risk-free security and one risky security.

$$P_1 = 1, \quad X_1 = (1, 1, 1); \qquad P_2 = 1, \quad X_2 = (1/2, 1, 2).$$

Suppose an investor has initial endowment 1 and von Neumann-Morgenstern utility function of the form $u(x) = x$. Furthermore, suppose that no short sale is allowed and the probability on each state is given by (1/3,1/3,1/3). Find

(a) the expected utility;

(b) the optimal portfolio;

(c) the maximal expected utility.

(3) Suppose there are one risk-free security and two risky security.

$$P_1 = 1, \quad X_1 = (2, 2, 2); \quad P_2 = 1, \quad X_2 = (1, 2, 4); \quad P_3 = 1, \quad X_3 = (1/2, 3, 2).$$

Suppose an investor has initial endowment 1 and von Neumann-Morgenstern utility function of the form $u(x) = x$. Furthermore, suppose that no short sale is allowed and the probability on each state is given by (1/4,1/4,1/2). Find

(a) the expected utility;

(b) the optimal portfolio;

(c) the maximal expected utility.