

CHAPTER 4

Preferences

4.1. Preference relations

Let \mathcal{X} be some non-empty set. An element $x \in \mathcal{X}$ will be interpreted as a possible choice of an economic agent.

Definition 4.1. A preference order (or preference relation) on \mathcal{X} is a binary relation \succ with the following two properties.

- (1) (Asymmetry) For $x, y \in \mathcal{X}$, if $x \succ y$, then $y \not\succ x$.
- (2) (Negative transitivity) If $x \succ y$ and $z \in \mathcal{X}$, then either $x \succ z$ or $z \succ y$ or both must hold.

Remark 4.2. (1) $y \succ x$ means that y is better than x .

- (2) Asymmetry means that the relations $x \succ y$ and $y \succ x$ cannot hold at the same time.
- (3) Negative transitivity means that the relations $x \succ y$, $y \succ z$, and $z \succ x$ cannot hold at the same time.
- (4) Not every relation can be a preference order, e.g., parents-children relation does not satisfy the negative transitivity condition.

Example 4.3. (1) Suppose \mathcal{X} = the collection of all fruits and \succ means your preference of the fruits, e.g.,

$$\text{Apple} \succ \text{Banana} \succ \text{Kiwi} \succ \text{Pineapple}.$$

(2) If $\mathcal{X} = \mathbb{R}$ and \succ is the usual strict inequality $>$. Then \succ is a preference order.

(3) Let $\mathcal{X} := [0, 1] \times [0, 1]$ and \succ is defined by

$$(x_1, x_2) \succ (y_1, y_2) \quad :\Longleftrightarrow \quad x_1 > y_1 \text{ and } x_2 > y_2$$

(see Figure 4.1). Then \succ is not a preference order, since the condition of negative transitivity fails, e.g., $(3/4, 3/4) \succ (1/2, 1/4)$, but the pair $(1/4, 1)$ does not have one of the preference relations $(3/4, 3/4) \succ (1/4, 1)$ and $(1/4, 1) \succ (1/2, 1/4)$.

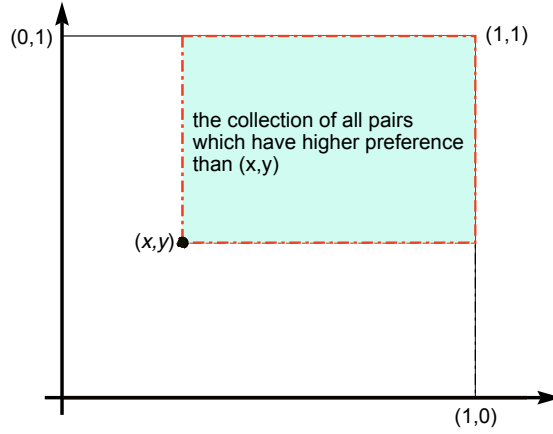


FIGURE 4.1. The collection of all pairs which have higher preference than (x, y)

(4) Let $\mathcal{X} := [0, 1] \times [0, 1]$ and \succ is defined by

$$(x_1, x_2) \succ (y_1, y_2)$$

$$:\Longleftrightarrow \quad \text{either } x_1 > y_1, \text{ or if } x_1 = y_1 \text{ and simultaneously } x_2 > y_2$$

for all $(x_1, x_2), (y_1, y_2) \in \mathcal{X}$ (see Figure 4.2). Then \succ is a preference order. This preference order is called the lexicographic preference relation.

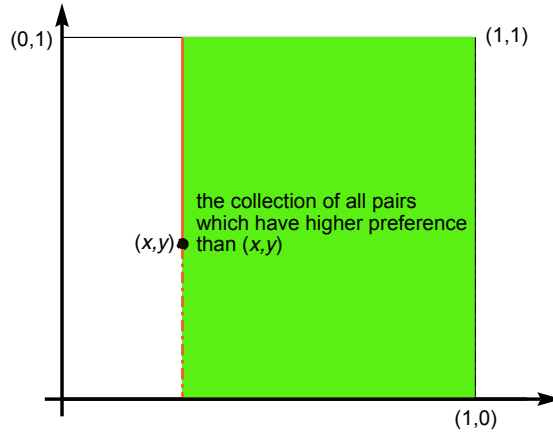


FIGURE 4.2. The collection of all pairs which have higher preference than (x, y)

Definition 4.4. (1) A preference order \succ on \mathcal{X} induces a corresponding weak preference order \succeq defined by

$$x \succeq y \quad :\Longleftrightarrow \quad y \not\succ x.$$

(2) A preference order \succ on \mathcal{X} induces a corresponding indifference relation \sim given by

$$x \sim y \quad :\Longleftrightarrow \quad x \succeq y \text{ and } y \succeq x.$$

Example 4.5. (1) Suppose \mathcal{X} = the collection of all colors and \succeq means your weak preference order of colors.

(2) As in Example 4.3 (2), \succeq is the usual inequality \geq and the indifference relation \sim is the identity $=$.

(3) As in Example 4.3 (4), then

$$\begin{aligned} (x_1, x_2) \succeq (y_1, y_2) \\ \Longleftrightarrow \quad \text{either } x_1 > y_1, \text{ or if } x_1 = y_1 \text{ and simultaneously } x_2 \geq y_2 \end{aligned}$$

for all $(x_1, x_2), (y_1, y_2) \in \mathcal{X}$.

Remark 4.6. (1) The asymmetry and the negative transitivity of \succ are equivalent to the following two respective properties of \succeq :

- (a) (Completeness) For all $x, y \in \mathcal{X}$, either $y \succeq x$ or $x \succeq y$ or both are true.
- (b) (Transitivity) If $x \succeq y$ and $y \succeq z$, then also $x \succeq z$.

- (2) Any complete and transitivity relation \succeq induces a preference order \succ via the negation of \succeq , i.e.,

$$y \succ x \quad :\Longleftrightarrow \quad x \not\succeq y.$$

- (3) The indifference relation \sim is an equivalent relation, i.e., the relation \sim satisfies reflexivity, transitivity and symmetry.
- (4) If $x \succ y \succeq z$, then $x \succ z$.

PROOF. Exercise. □

Remark 4.7. Some problems arise in the preference relation:

- (1) An individual's preferences may fail to satisfy the transitivity property for a number of reasons. One difficulty arises because of the problem of just perceptible differences. For example, if we ask an individual to choose between two similar shades of gray for painting her room, she may be unable to tell the difference between the colors and will therefore be indifferent. Suppose now that we offer her a choice between the lighter of two gray paints and a slighter shade. She may again be unable to tell the difference. If we continue in this fashion, letting the paint colors get progressively lighter with each successive choice experiment, she may express indifference at each step. Yet, if we offer her a choice between the original (darkest) shade and the final (almost white) color, she would be able to

distinguish between the colors and is likely to prefer one of them. This, however, violates transitivity.

- (2) Another potential problem arises when the manner in which alternatives are presented matters for choice. This is known as the framing problem. For example, imagine that you are about to purchase a stereo for NT \$ 7000 and a calculator for NT \$ 500. The salesman tells you that the calculator is on sale for NT \$ 150 less at the other branch of the store, located 20 minutes away. The stereo is the same price there. Would you make the trip to the other store?

It turns out that the fraction of respondents saying that they would travel to the other store for NT \$ 150 discount is much higher than the fraction who say they would travel when the question is changed so that the NT \$ 150 saving is on the stereo.

- (3) Formulation of the question: Imagine that the U.S. is preparing for the outbreak of an unusual Asian disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed. Assume that the exact scientific estimates of the consequences of the programs are as follows:

(A) 200 people will be saved.

(B) There is one-third probability that 600 people will be saved and a two-thirds probability that no people will be saved.

Which of the two programs would you favor? The most will choose (A) (72 %). A clear majority of respondents prefer saving 200 lives for sure over a gamble that offers a one-third chance of saving 600 lives. Now consider another problem in which the same cover story is followed by a different description of the prospects associated with the two programs:

(C) 400 people will die.

(D) There are a one-third probability that nobody will die and a two thirds probability that 600 people will die.

The most respondents prefer (D) (78%) than (C). It is easy to verify that options (C) and (D) are undistinguishable in real terms from options (A) and (B), respectively.

(4) It is often the case that apparently intransitive behavior can be explained fruitfully as the result of the intersection of several primitive preferences. For example, a household formed by Mom (M), Dad (D), and Child (C) makes decisions by majority voting. The alternatives for Friday evening entertainment are attending an opera (O), a rock concert (R), or an ice-skating show (I). There are three numbers of the household have the individual preferences:

$$O \succ_M R \succ_M I, \quad I \succ_D O \succ_D R, \quad R \succ_C I \succ_C O,$$

where $\succ_M, \succ_D, \succ_C$ are the transitive individual preference relations. Now imagine three majority-rule votes: O versus R, R versus I, and I versus O. The result of these votes will make the household's preferences \succ have the intransitive form: $O \succ R \succ I \succ O$. This is called the Condorcet paradox, and it is a central difficulty for the theory of group decision making.

4.2. Numerical representation

Definition 4.8. A numerical representation of a preference order \succ is a function $U : \mathcal{X} \longrightarrow \mathbb{R}$ such that

$$y \succ x \quad \Longleftrightarrow \quad U(y) > U(x). \quad (4.1)$$

Example 4.9. (1) As in Example 4.3(1). Suppose \mathcal{X} = the collection of all fruits

and

$$\text{Apple} \succ \text{Banana} \succ \text{Kiwi} \succ \text{Pineapple}.$$

We may set

$$U(\text{Apple}) = 100, \quad U(\text{Banana}) = 50, \quad U(\text{Kiwi}) = 1, \quad U(\text{Pineapple}) = -200000.$$

(2) Let \mathcal{X} = the collection of all human being and define a preference order on \mathcal{X} by

$$y \succ x \iff x \text{ is shorter than } y.$$

Let $U(x)$ = the height of x , then

$$y \succ x \iff U(y) > U(x),$$

i.e., U is a numerical representation of \succ .

(3) Suppose \mathcal{X} is the collection of all countries in the world.

(a) If $x \succ y$ means that x has more population than y , we may define a numerical representation of \succ by

$$U(x) = \text{the population in the country } x.$$

(b) If $x \succ y$ means that x is smaller than y , we may define a numerical representation of \succ by

$$U(x) = - \text{the area of } x.$$

(c) If $x \succ y$ means that the proportion of males and females in the country x closer to $1/2$ than that of y , then we may define a numerical representation of \succ by

$$U(x) = \frac{1}{2} - \left| \frac{1}{2} - \frac{\text{the number of males in the country } x}{\text{total population in the country } x} \right|.$$

Remark 4.10. (1) (4.1) is equivalent to

$$y \succeq x \quad \Longleftrightarrow \quad U(y) \geq U(x).$$

This implies that

$$y \sim x \quad \Longleftrightarrow \quad U(y) = U(x).$$

(2) The numerical representation U is not unique. If f is any strictly increasing function, then $\tilde{U}(x) := f(U(x))$ is again a numerical representation.

Example 4.11. As in Example 4.9 (2), let

$$\tilde{U}(x) = 10 \times (\text{the height of } x)^2 + 200,$$

then

$$y \succ x \quad \Longleftrightarrow \quad U(y) > U(x) \quad \Longleftrightarrow \quad \tilde{U}(y) > \tilde{U}(x).$$

Question: Does every preference order have a numerical representation?

Answer: No! See the below example.

Example 4.12. Let $\mathcal{X} := [0, 1] \times [0, 1]$ and \succ is a preference order on \mathcal{X} defined by

$$(x_1, x_2) \succ (y_1, y_2) \quad :\Longleftrightarrow \quad \text{either } x_1 > y_1, \text{ or if } x_1 = y_1 \text{ and simultaneously } x_2 > y_2.$$

Claim: The preference order \succ has no numerical representation.

Suppose that \succ admits a numerical representation U . Set

$$d(\alpha) := U(\alpha, 1) - U(\alpha, 0).$$

Then $d(\alpha) > 0$ for all $\alpha \in [0, 1]$. Hence,

$$[0, 1] = \bigcup_{n=1}^{\infty} \left\{ \alpha \in [0, 1] : d(\alpha) > \frac{1}{n} \right\}.$$

Denote $A_n = \left\{ \alpha \in [0, 1] : d(\alpha) > \frac{1}{n} \right\}$. Then there must be one set A_{n_0} having infinitely many elements. For any $N \in \mathbb{N}$, choose $\alpha_1, \alpha_2, \dots, \alpha_N \in A_{n_0}$ such that $\alpha_1 < \dots < \alpha_N$. Since $(\alpha_{i+1}, 0) \succ (\alpha_i, 1)$,

$$U(\alpha_{i+1}, 0) - U(\alpha_i, 0) > U(\alpha_i, 1) - U(\alpha_i, 0) = d(\alpha_i) > \frac{1}{n_0}.$$

Hence,

$$\begin{aligned} U(1, 1) - U(0, 0) &= \underbrace{U(1, 1) - U(\alpha_N, 0)}_{>0} + \sum_{i=1}^{N-1} \underbrace{(U(\alpha_{i+1}, 0) - U(\alpha_i, 0))}_{>1/n_0} + \underbrace{U(\alpha_1, 0) - U(0, 0)}_{>0} \\ &= \frac{N-1}{n_0}. \end{aligned}$$

Since N is arbitrary, $U(1, 1) - U(0, 0)$ must be infinite, which is impossible.

Question: When does a preference order \succ have a numerical representation?

Definition 4.13. Let \succ be a preference order on \mathcal{X} . A subset \mathcal{Z} of \mathcal{X} is called order dense if for any pair $x, y \in \mathcal{X}$ such that $x \succ y$, there exists some $z \in \mathcal{Z}$ with $x \succeq z \succeq y$.

We can regard the order dense as the usual definition of the dense subset. So it is easier to understand what it means.

Theorem 4.14. *A preference order \succ admits a numerical representation on \mathcal{X} if and only if \mathcal{X} admits a countable order dense subset \mathcal{Z} . In particular, any preference order admits a numerical representation if \mathcal{X} is countable.*

Example 4.15. (1) If $\mathcal{X} = \mathbb{R}$ and \succ is the usual strict inequality $>$. Then there exists a countable order subset of \mathbb{R} , e.g., \mathbb{Q} .

- (2) If $\mathcal{X} = [0, 1] \times [0, 1]$ and \succ is given in Example 4.12. Then \mathcal{X} contains no countable order dense subset. Note that $(\mathbb{Q} \cap [0, 1]) \times (\mathbb{Q} \cap [0, 1])$ is not a countable order dense subset of $[0, 1] \times [0, 1]$, since $(\pi/4, 1) \succ (\pi/4, 0)$, but there does not exist any pair $(x, y) \in (\mathbb{Q} \cap [0, 1]) \times (\mathbb{Q} \cap [0, 1])$ such that

$$(\pi/4, 1) \succ (x, y) \succ (\pi/4, 0).$$

- (3) If $\mathcal{X} = [0, 1] \times [0, 1]$ and \succ is defined by

$$(x_1, x_2) \succ (y_1, y_2) \quad :\Longleftrightarrow \quad x_1 > y_1.$$

Then \mathcal{X} admits a numerical representation, since $(\mathbb{Q} \cap [0, 1]) \times \{0\}$ is a countable order dense subset of $[0, 1] \times [0, 1]$.

PROOF OF THEOREM 4.14. “ \Leftarrow ” Suppose there exists a countable order dense subset \mathcal{Z} of \mathcal{X} . For $x \in \mathcal{X}$, set

$$\bar{\mathcal{Z}}(x) := \{z \in \mathcal{Z} : z \succ x\} \quad \text{and} \quad \underline{\mathcal{Z}}(x) := \{z \in \mathcal{Z} : x \succ z\}.$$

For $x, y \in \mathcal{X}$,

- (1) if $x \succeq y$, then $\bar{\mathcal{Z}}(x) \subseteq \bar{\mathcal{Z}}(y)$ and $\underline{\mathcal{Z}}(x) \supseteq \underline{\mathcal{Z}}(y)$;
- (2) if $x \succ y$, then at least one of the conclusion in (1) is strict. Since \mathcal{Z} is order dense subset of \mathcal{X} , there exists $z \in \mathcal{Z}$ such that either $x \succ z \succeq y$ or $x \succeq z \succ y$. In the first case, $z \in \underline{\mathcal{Z}}(x) \setminus \underline{\mathcal{Z}}(y)$, and in the second case, $z \in \bar{\mathcal{Z}}(y) \setminus \bar{\mathcal{Z}}(x)$.

Take a strictly positive probability distribution μ on \mathcal{Z} , e.g., for $\mathcal{Z} = \{z_1, z_2, \dots, z_n, \dots\}$, we may take $\nu(z_n) = \frac{6}{\pi^2 n^2}$. Let

$$U(x) := \sum_{z \in \underline{\mathcal{Z}}(x)} \mu(z) - \sum_{z \in \bar{\mathcal{Z}}(x)} \mu(z).$$

By the above argument, we have $U(x) > U(y)$ if and only if $x \succ y$ and U is the desired numerical representation.

“ \implies ” Suppose U is a numerical representation.

Original Thought: Consider $U^{-1}(\mathbb{Q})$. But the set $U^{-1}(\mathbb{Q})$ may neither countable nor order dense. Hence, we have to make some rearrangement.

Let \mathcal{J} be the countable set

$$\mathcal{J} : \{[a, b] : a, b \in \mathbb{Q}, a < b, U^{-1}([a, b]) \neq \emptyset\}.$$

For every interval $I \in \mathcal{J}$ we can choose some $z_I \in \mathcal{X}$ with $U(z_I) \in I$. Define

$$A := \{z_I : I \in \mathcal{J}\}.$$

However, this set A does not guaranteed the property of the order dense condition.

Let

$$C := \{(x, y) : x, y \in \mathcal{X} \setminus A, y \succ x \text{ and there does not exist } z \in A \text{ with } y \succ z \succ x\}.$$

In fact,

$$C := \{(x, y) : x, y \in \mathcal{X} \setminus A, y \succ x \text{ and there does not exist } \bar{z} \in \mathcal{X} \text{ with } y \succ \bar{z} \succ x\}.$$

Otherwise, there exists $\bar{z} \in \mathcal{X}$ such that $y \succ \bar{z} \succ x$. This implies that $U(x) < U(\bar{z}) < U(y)$. Thus, we can find $a, b \in \mathbb{Q}$ such that

$$U(x) < a < U(\bar{z}) < b < U(y),$$

so the interval $[a, b] \in \mathcal{J}$. Then there exists $z \in A$ with $y \succ z \succ x$, contradicts to the assumption that $(x, y) \in C$.

It follows that all intervals $(U(x), U(y))$ with $(x, y) \in C$ are disjoint and non-empty. Hence, there are only countably many such intervals. For each such interval choose exactly $(x, y) \in C$ such that $U(x)$ and $U(y)$ are the endpoints of the interval. Set B as the collection of all such endpoints.

Claim: The set $\mathcal{Z} := A \cup B$ is a countable order dense subset of \mathcal{X} .

- (1) Since both of A and B are countable, \mathcal{Z} is countable.
- (2) If $x, y \in \mathcal{X} \setminus \mathcal{Z}$ with $y \succ x$, then either there is some $z \in A$ such that $y \succ z \succ x$, or $(x, y) \in C$. In the latter case, there will be some $z \in B$ with $U(y) = U(z) > U(x)$ and, consequently, $y \succeq z \succ x$. Hence, \mathcal{Z} is an order dense subset of \mathcal{X} .

□

4.3. Continuity*

Question: When is the function U continuous?

Since the domain \mathcal{X} of the function U may not be \mathbb{R} or \mathbb{R}^n , even may not be a metric space, e.g. \mathcal{X} is the collection of all colors. Hence, we have to use another method to define the continuity of a function. In advanced calculus, it has been shown that a function f is continuous if and only if its pre-image of an open set is still open. However, since there may not be a metric in \mathcal{X} , we cannot define open sets in a similar way as in the Euclidean space. How can we define a continuous function in this setting? In a space without metric, we may use the following method to define an open set.

Definition 4.16. (1) A topology on a set \mathcal{X} is a collection \mathcal{F} of subsets of \mathcal{X} having the following properties:

- (a) \emptyset and \mathcal{X} are in \mathcal{F} .
- (b) The union of the elements of any subcollection of \mathcal{F} is in \mathcal{F} .
- (c) The intersection of the elements of any finite subcollection of \mathcal{F} is in \mathcal{F} .
- (2) A set \mathcal{X} for which a topology \mathcal{F} has been specified is called a topological space.
- (3) If \mathcal{X} is a topological space with topology \mathcal{F} , we say that a subset U of \mathcal{X} is an open set of \mathcal{X} if U belongs to the collection \mathcal{F} .

Hence, we may define open sets in a space without metric. Since every metric space is in a natural way a topological space, the topological definition of open sets generalises the metric space definition. Moreover, we may define a closed set as a set whose complement is open.

Example 4.17. (1) Let \mathcal{X} be a three-element set, $\mathcal{X} = \{a, b, c\}$. There are many possible topologies on \mathcal{X} . For example,

- (a) $\{\emptyset, \mathcal{X}\}$;
- (b) $\{\emptyset, \mathcal{X}, \{b\}\}$;
- (c) $\{\emptyset, \mathcal{X}, \{a, b\}\}$;
- (d) $\{\emptyset, \mathcal{X}, \{a\}, \{b, c\}\}$;
- (e) $\{\emptyset, \mathcal{X}, \{a\}, \{a, b\}\}$;
- (f) $\{\emptyset, \mathcal{X}, \{a\}, \{b\}, \{a, b\}\}$;
- (g) $\{\emptyset, \mathcal{X}, \{b\}, \{a, b\}, \{b, c\}\}$;
- (h) $\{\emptyset, \mathcal{X}, \{b\}, \{a, b\}, \{b, c\}, \{c\}\}$;
- (i) $\{\emptyset, \mathcal{X}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$;

all of them are the topologies on \mathcal{X} . We can see that even a three-element set has many different topologies. But not every collection of subsets of \mathcal{X} is a topology on \mathcal{X} , for instance, the collections of sets

- (a) $\{\emptyset, \mathcal{X}, \{a\}, \{b\}\}$;
- (b) $\{\emptyset, \mathcal{X}, \{a, b\}, \{b, c\}\}$

are not topologies on \mathcal{X} .

- (2) If \mathcal{X} is any non-empty set, the collection of all subsets of \mathcal{X} is a topology on \mathcal{X} ; it is called the discrete topology. The collection consisting of \mathcal{X} and \emptyset only is also a topology on \mathcal{X} ; we shall call it the indiscrete topology, or the trivial topology.
- (3) Let \mathcal{X} be a set; and let \mathcal{T}_f be the collection of all subsets U of \mathcal{X} such that $\mathcal{X} \setminus U$ is either finite or is all of \mathcal{X} . Then \mathcal{T}_f is a topology on \mathcal{X} , called the finite complement topology.
- (4) Let \mathcal{X} be a set; and let \mathcal{T}_c be the collection of all subsets U of \mathcal{X} such that $\mathcal{X} \setminus U$ is either countable or is all of \mathcal{X} . Then \mathcal{T}_c is a topology on \mathcal{X} .

Remark 4.18. Every metric space can be given a metric topology, in which the basic open sets are open balls defined by the metric. This is the standard topology on any normed vector space. On a finite-dimensional vector space this topology is the same for all norms. Whenever we consider a metric space, we shall suppose that it is given the metric topology unless we specially state otherwise.

Definition 4.19. Let \mathcal{X} be a topological space. A preference order \succ is called continuous if for all $x \in \mathcal{X}$

$$\bar{\mathcal{B}}(x) := \{y \in \mathcal{X} : y \succ x\} \quad \text{and} \quad \underline{\mathcal{B}}(x) := \{y \in \mathcal{X} : x \succ y\}$$

are open subsets of \mathcal{X} .

Remark 4.20. Every preference order that admits a continuous numerical representation is continuous relative to the same topology (using the definition of continuous function). But the converse is not true in general.

Remark 4.21. (1) Let \mathcal{X} be a topological space, we may use the weak preference order to consider the continuity of the numerical representation. Hence, a weak preference order is called continuous if for all $x \in \mathcal{X}$

$$\{y \in \mathcal{X} : y \succeq x\} \quad \text{and} \quad \{y \in \mathcal{X} : x \succeq y\}$$

are closed subsets of \mathcal{X} .

(2) If \mathcal{X} is a metric space, we may define the weak preference order in another way.

We say the weak preference order \succeq on \mathcal{X} is continuous if it is preserved under limits. That is, for any sequence of pairs $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ with $x_n \succeq y_n$ for all n , we have

$$\lim_{n \rightarrow \infty} x_n \succeq \lim_{n \rightarrow \infty} y_n.$$

Example 4.22. (1) As in Example 4.12 the preference order is not continuous; if $(x_1, x_2) \in [0, 1] \times [0, 1]$ is given, then

$$\{(y_1, y_2) : (y_1, y_2) \succ (x_1, x_2)\} = ((x_1, 1] \times [0, 1]) \cup (\{x_1\} \times (x_2, 1])$$

which is “typically” not an open subset of $[0, 1] \times [0, 1]$. But this set is open relative to the discrete topology.

(2) Consider four points on \mathbb{R}^2 : $A(0, 0)$, $B(1, 0)$, $C(0, 1)$, and $D(2, 2)$. Define a preference order \succ on \mathbb{R}^2 by

$$P(x_1, x_2) \succ Q(y_1, y_2) \\ \iff |AP|^2 + |BP|^2 + |CP|^2 + |DP|^2 < |AQ|^2 + |BQ|^2 + |CQ|^2 + |DQ|^2.$$

Then

$$\{(x, y) : (x, y) \succ (1, 1)\} = \left\{ (x, y) : \left(x - \frac{3}{4}\right)^2 + \left(y - \frac{3}{4}\right)^2 < \frac{1}{8} \right\},$$

which is clearly open on \mathbb{R}^2 .

- Definition 4.23.** (1) A topological space \mathcal{X} is called a topological Hausdorff space if any two distinct points in \mathcal{X} have disjoint open neighborhood.
- (2) A space having a countable (order) dense subset is said to be separable.
- (3) The topological space \mathcal{X} is called connected if \mathcal{X} cannot be written as the union of two disjoint and non-empty open sets. Explicitly, \mathcal{X} is said to be connected if there does not exist two open sets A, B such that

$$A \cap \mathcal{X} \neq \emptyset, \quad B \cap \mathcal{X} \neq \emptyset, \quad A \cap B = \emptyset, \quad \text{and} \quad \mathcal{X} \subseteq A \cup B.$$

Theorem 4.24. *Suppose \mathcal{X} is a topological Hausdorff space and \succ is a preference order on \mathcal{X} . Then the following properties are equivalent:*

- (1) \succ is continuous.
- (2) The set $\{(x, y) : y \succ x\}$ is open in $\mathcal{X} \times \mathcal{X}$.
- (3) The set $\{(x, y) : y \succeq x\}$ is closed in $\mathcal{X} \times \mathcal{X}$.

PROOF. See Föllmer-Schied [3] P. 47. □

Theorem 4.25. *Let \mathcal{X} be a topological space which is separable and connected. Then every continuous preference order on \mathcal{X} admits a continuous numerical representation.*

PROOF. See Debreu [2], Proposition 3 and 4. □

4.4. Von Neumann-Morgenstern representation

In this section, the set \mathcal{X} can be identified with a subset \mathcal{M} of the collection of all probability distributions on a measurable space (S, \mathcal{S}) .

Notation 4.26. The elements of \mathcal{M} are sometimes called lotteries.

Definition 4.27. A numerical representation U of a preference order \succ on \mathcal{M} is called a von Neumann-Morgenstern representation if there exists a real function u on S such that

$$U(\mu) = \int_S u(x) \mu(dx), \quad \text{for all } \mu \in \mathcal{M}. \quad (4.2)$$

Remark 4.28. (1) If μ has a cumulative distribution function F , then (4.2) can be written as

$$U(\mu) = \int_S u(x) dF(x).$$

(2) If μ has a probability density function p , then (4.2) can be written as

$$U(\mu) = \int_S u(x)p(x) dx.$$

(3) If S has only finite elements, i.e., μ has a probability mass function m , then

$$U(\mu) = \sum_{x \in S} u(x)m(x).$$

Remark 4.29. Any von Neumann-Morgenstern representation U is affine on \mathcal{M} in the sense that

$$U(\alpha\mu + (1 - \alpha)\nu) = \alpha U(\mu) + (1 - \alpha)U(\nu)$$

for all $\mu, \nu \in \mathcal{M}$ and $\alpha \in [0, 1]$.

Question: When has a preference order a von Neumann-Morgenstern representation?

Definition 4.30. (1) A preference order \succ on \mathcal{M} satisfies the independent axiom if, for all $\mu, \nu \in \mathcal{M}$, the relation $\mu \succ \nu$ implies

$$\alpha\mu + (1 - \alpha)\lambda \succ \alpha\nu + (1 - \alpha)\lambda$$

for all $\lambda \in \mathcal{M}$ and all $\alpha \in (0, 1]$.

- (2) A preference order \succ on \mathcal{M} satisfies the Archimedean axiom if any triple $\mu \succ \lambda \succ \nu$ there are $\alpha, \beta \in (0, 1)$ such that

$$\alpha\mu + (1 - \alpha)\nu \succ \lambda \succ \beta\mu + (1 - \beta)\nu.$$

Remark 4.31. (1) The independent axiom is also called the substitution axiom.

Regarding $\alpha\mu + (1 - \alpha)\lambda$ as a two step procedure. If we mix two lotteries with a third one, then the preference order of the two resulting mixtures is independent of the particular third lottery used.

- (2) The Archimedean axiom is also called the continuity axiom.

$$\alpha\mu + (1 - \alpha)\nu \longrightarrow \mu \quad \text{as } \alpha \rightarrow 1$$

$$\alpha\mu + (1 - \alpha)\nu \longrightarrow \nu \quad \text{as } \alpha \rightarrow 0.$$

The small changes in probabilities do not change the nature of the ordering between the lotteries.

Example 4.32. (1) In the game of showhand,

$$\begin{array}{ll} \text{the first card} & A \succ K \\ \text{the second card} & \frac{1}{2}A + \frac{1}{2}K \prec \frac{1}{2}K + \frac{1}{2}K. \end{array}$$

This does not satisfy the independent axiom.

- (2) Suppose

μ : win 100 with probability 1,

ν : win 10 with probability 1,

λ : die with probability 1.

Then $\mu \succ \nu \succ \lambda$, but for any $\alpha \in (0, 1)$, $\alpha\mu + (1 - \alpha)\lambda \not\succ \nu$. Hence the continuity axiom is not satisfied.

Proposition 4.33. *The affinity of U implies the independent axiom and Archimedean axiom.*

PROOF. (1) If $\mu \succ \nu$, then $U(\mu) > U(\nu)$. Thus,

$$\begin{aligned} U(\alpha\mu + (1 - \alpha)\lambda) &= \alpha U(\mu) + (1 - \alpha)U(\lambda) > \alpha U(\nu) + (1 - \alpha)U(\lambda) \\ &= U(\alpha\nu + (1 - \alpha)\lambda). \end{aligned}$$

Thus implies that $\alpha\mu + (1 - \alpha)\lambda \succ \alpha\nu + (1 - \alpha)\lambda$.

(2) If $\mu \succ \lambda \succ \nu$, then $U(\mu) > U(\lambda) > U(\nu)$. Then there exists $\alpha, \beta \in (0, 1)$ such that

$$\alpha U(\mu) + (1 - \alpha)U(\nu) > U(\lambda) > \beta U(\mu) + (1 - \beta)U(\nu).$$

Thus,

$$U(\alpha\mu + (1 - \alpha)\nu) > U(\lambda) > U(\beta\mu + (1 - \beta)\nu).$$

This implies $\alpha\mu + (1 - \alpha)\nu \succ \lambda \succ \beta\mu + (1 - \beta)\nu$.

□

Theorem 4.34. *Suppose that \succ is a preference order on \mathcal{M} satisfying both of the Archimedean and the independent axiom. Then there exists an affine numerical representation U of \succ . Moreover, U is unique up to positive affine transformation, i.e., any other numerical representation \tilde{U} with these properties is of the form $\tilde{U} = aU + b$ for some $a > 0$ and $b \in \mathbb{R}$.*

PROOF. See Föllmer-Schied [3] P. 54.

□

Definition 4.35. A probability measure is called a simple probability measure if there exist $x_1, \dots, x_N \in S$ and $\alpha_1, \dots, \alpha_N \in [0, 1]$ with $\sum_{i=1}^N \alpha_i = 1$ such that

$$\mu = \sum_{i=1}^N \alpha_i \delta_{x_i},$$

where δ_x is the Dirac measure, i.e.,

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Corollary 4.36. Suppose that \mathcal{M} is the set of all simple distributions on S , i.e.,

$$\mathcal{M} = \left\{ \alpha_1 \delta_{x_1} + \dots + \alpha_N \delta_{x_N} : \alpha_i > 0 \text{ and } x_i \in S \text{ for all } i \text{ with } \sum_{i=1}^N \alpha_i = 1 \right\}.$$

Suppose that \succ is a preference order on \mathcal{M} that satisfies both the Archimedean and the independent axiom. Then there exists a von Neumann-Morgenstern representation U . Moreover, both U and u are unique to positive affine transformation.

PROOF. Suppose that U is an affine numerical representation. Let $U(x) := U(\delta_x)$ for all $x \in S$ and $\mu \in \mathcal{M}$ of the form

$$\mu = \alpha_1 \delta_{x_1} + \dots + \alpha_N \delta_{x_N}$$

with $\sum_{i=1}^N \alpha_i = 1$. Due to the affinity of U , we have

$$U(\mu) = U\left(\sum_{i=1}^N \alpha_i \delta_{x_i}\right) = \sum_{i=1}^N \alpha_i U(\delta_{x_i}) = \sum_{i=1}^N \alpha_i u(x_i) = \int u(x) \mu(dx).$$

Thus, U is a von Neumann-Morgenstern representation. □

Corollary 4.37. Let \mathcal{M} be the set of all probability distributions on a finite set S and let \succ be a preference order on \mathcal{M} satisfying the independent axiom and the Archimedean axiom. Then there exists a von Neumann-Morgenstern representation.

PROOF. Exercise. □

Example 4.38. Let \mathcal{M} be the collection of probability measures on $S := \{1, 2, \dots\}$ for which $U(\mu) := \lim_{k \uparrow \infty} k\mu(k)$ exists and is finite. Then U is affine and induces a preference order on \mathcal{M} satisfying both of the Archimedean axiom and the independent axiom. However, U does not admit a von Neumann-Morgenstern representation.

4.5. Paradox and examples

Example 4.39. (Allais paradox) The so-called Allais paradox questions the descriptive aspect of expected utility by considering the following lotteries. Lottery ν_1 is given by

$$\nu_1 = 0.33 \delta_{2500} + 0.66 \delta_{2400} + 0.01 \delta_0.$$

Lottery μ_1 is given by

$$\mu_1 = \delta_{2400}.$$

When asked, most people (82%) prefer the sure amount, even though lottery ν_1 has the larger expected value 2409.

Next, consider the following two lotteries μ_2 and ν_2 :

$$\mu_2 = 0.34 \delta_{2400} + 0.66 \delta_0, \quad \text{and} \quad \nu_2 = 0.33 \delta_{2500} + 0.67 \delta_0.$$

Here people tend to prefer the slightly riskier lottery ν_2 over μ_2 (83%), in accordance with the expectations of ν_2 and μ_2 , which are 825 and 816, respectively.

This means that at least 65% choose both $\mu_1 \succ \nu_1$ and $\nu_2 \succ \mu_2$. As pointed out by M. Allais, this simultaneous choice leads to a “paradox” in the sense that it is inconsistent with the von Neumann-Morgenstern paradigm. More precisely, any preference relation

\succ for which $\mu_1 \succ \nu_1$ and $\nu_2 \succ \mu_2$ are both valid violates the independent axiom. If the independent axiom were satisfied, then

$$\alpha\mu_1 + (1 - \alpha)\nu_2 \succ \alpha\nu_1 + (1 - \alpha)\nu_2 \succ \alpha\nu_1 + (1 - \alpha)\mu_2$$

for all $\alpha \in (0, 1)$. By taking $\alpha = 1/2$ we would have

$$\frac{1}{2}(\mu_1 + \nu_2) \succ \frac{1}{2}(\nu_1 + \mu_2)$$

which is a contradiction to the fact that

$$\frac{1}{2}(\mu_1 + \nu_2) = \frac{1}{2}(\nu_1 + \mu_2).$$

Therefore, the independent axiom was violated by at least 65% of the people who were interviewed.

The Allais paradox is a classical example of decision making that runs counter to the expected utility optimization. In Chapter 9 we shall introduce a model that have been developed in order to avoid these violations.

Remark 4.40. In some textbooks there is an alternative version of the Allais paradox. Consider the following two experiments.

- A choice is offered between A and B:

$$\begin{array}{ll} A & : \text{ win \$1 million with probability 1} \\ B & : \left\{ \begin{array}{ll} \text{win \$1 million} & \text{with probability 0.89} \\ \text{win \$5 million} & \text{with probability 0.1} \\ \text{win \$0} & \text{with probability 0.01} \end{array} \right. \end{array}$$

- A choice is offered between C and D:

$$\begin{aligned} C &: \begin{cases} \text{win \$1 million} & \text{with probability 0.11} \\ \text{win \$0} & \text{with probability 0.89} \end{cases} \\ D &: \begin{cases} \text{win \$5 million} & \text{with probability 0.1} \\ \text{win \$0} & \text{with probability 0.9} \end{cases} \end{aligned}$$

Results show that in Part 1, most investors choose A, and in part 2, most investors choose D. These decisions are inconsistent and contradict the expected utility theory.

The preference of A over B implies (all figures in million dollars):

$$1u(1) > 0.01u(0) + 0.89u(1) + 0.1u(5), \quad (4.3)$$

and the preference of D over C implies

$$0.9u(0) + 0.1u(5) > 0.89u(0) + 0.11u(1). \quad (4.4)$$

Inequality (4.3) can be written as

$$0.01u(0) + 0.1u(5) < 0.11u(1),$$

and inequality (4.4) can be rewritten as

$$0.01u(0) + 0.1u(5) > 0.11u(1).$$

This outcome is inconsistent: it suggests either that subjects do not maximize expected utility or that the expected utility model needs to be modified in order to accommodate and explain paradoxical results such as these.

Example 4.41. This is an example illustrate another example of mental accounting in which the posting of a cost to an account is controlled by topical organization:

- (A) Imagine that you have decide to see a play and paid the admission price of NT \$ 300 per ticket. As you enter the theater, you discover that you have lost the ticket. The seat was not marked, and the ticket cannot be recovered.
- Would you pay NT \$ 300 for another ticket?

Yes: 46% No: 54%.

- (B) Imagine that you have decide to see a play and paid the admission price of NT \$ 300 per ticket. As you enter the theater, you discover that you have lost NT \$ 300 bill.
- Would you pay NT \$ 300 for another ticket?

Yes: 88% No: 12%.

The difference between the responses to the two problems is intriguing. Why are so many people unwilling to spend NT \$ 300 after having lost a ticket, if they would readily spend that sum after losing an equivalent amount of cash? We tribute the difference to the topical organization of mental accounts. Going to the theater is normally viewed as a transaction in which the cost of the ticket is exchanged for the experience of seeing play. Buying a second ticket increases the cost of seeing the play to level that many respondents apparently find unacceptable. In contrast, the lost of the cash is not posted to the account of the play, and it affects the purchase of a ticket only by making the individual feel slightly less affluent.

4.6. Exercise

- (1) Find a preference relations for the following sets \mathcal{X} .
- (a) \mathcal{X} = the collection of all baseball teams in the world.

- (b) \mathcal{X} = the collection of all classmates in the course.
 - (c) \mathcal{X} = the collection of all cities in the world.
 - (d) \mathcal{X} = the collection of all books.
 - (e) $\mathcal{X} = \mathbb{R}^2$.
 - (f) $\mathcal{X} = \mathbb{N}^2$.
- (2) Find a corresponding numerical representation for each preference relation given in the last problems if it exists.
- (3) Let \mathcal{X} be a non-empty set and $x, y, z \in \mathcal{X}$.
- (a) If $x \succ y$ and $y \succeq z$, prove that $x \succ z$.
 - (b) If $x \succ y$ and $y \succ z$, prove that $x \succ z$.
- (4) Prove that the indifference relation \sim is an equivalent relation, i.e., the relation \sim satisfies the conditions of reflexivity, symmetry, and transitivity.
- (5) (a) Suppose that the binary relation \succ is a preference order, i.e., \succ satisfies the conditions of asymmetry and negative transitivity, prove that the binary relation \succeq defined by

$$x \succeq y \quad \Longleftrightarrow \quad y \not\succ x,$$

satisfies the conditions of completeness and transitivity.

- (b) Suppose the binary relation \succeq satisfies the conditions of completeness and transitivity, show that the binary relation defined by

$$x \succ y \quad \Longleftrightarrow \quad y \not\succeq x,$$

is a preference order, i.e., \succ satisfies the conditions of asymmetry and negative transitivity.

- (6) Let $\mathcal{X} = [0, 1] \times [0, 1]$

- (a) Find a preference order \succ on \mathcal{X} such that it admits a numerical representation.
 - (b) Find the numerical representation of \succ .
 - (c) Find the countable order dense subset of \mathcal{X} .
- (7) Consider a set

$$\begin{aligned}\mathcal{X} &= \{0, 1\} \times \{0, 1\} \times \cdots = \{0, 1\}^\infty \\ &= \text{the collection of all infinite sequences } (a_n)_{n \in \mathbb{N}} \text{ with } a_n = 0 \text{ or } 1,\end{aligned}$$

and a binary relation on \mathcal{X} defined by

$$\begin{aligned}(a_n) &\succ (b_n) \\ \iff &\text{“}a_1 > b_1\text{” or “for some } m > 1, a_i = b_i \text{ for all } i < m \text{ and } a_m > b_m\text{”}.\end{aligned}$$

- (a) Show that \mathcal{X} contains uncountably many elements.
- (b) Show that \succ is a preference order on \mathcal{X} .
- (c) Find a numerical representation of \succ .
- (d) Find a countable order dense subset of \mathcal{X} .