CHAPTER 1

Arbitrage and Pricing

1.1. Security markets

Consider a one period model: $t = 0, 1$, where

$t = 0$: now

$t = 1$: fixed time point in the future (e.g., 1 hour, 1 day, 1 month, 1 year, ...)

Securities are traded at time 0; payoffs are realized at time 1.

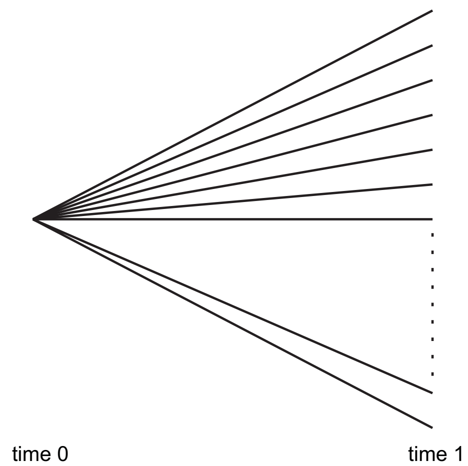

FIGURE 1.1. S states, S possible outcomes

Some basic question:

(1) Price at time 1?

(2) How to invest? In other words, we want to study portfolio optimization, e.g., suppose there are two assets with prices 100 and 50 at time 0, respectively.

the wealth at time 0: $100a + 50b = \text{constant} = \text{initial endowment}$

the wealth at time 1: $aX_1(w) + bX_2(w) \stackrel{!}{=} \max.$

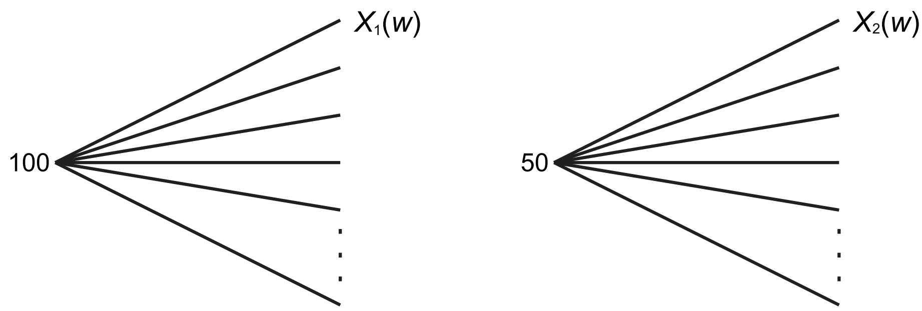

FIGURE 1.2

In this chapter we aim to find the relation of the prices at time 0 and payoff at time 1,

Suppose that there are J securities in the market: At time 1, the security j with payoff $X_{ji}, X_{j2}, ..., X_{jS}$ at each state, where X_{js} means the payoff of one share of security j in state s at time 1.

Denote $X_j = (X_{ji}X_{j2}, \cdots X_{jS}) \in \mathbb{R}^S$.

Definition 1.1. Let

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_J \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1S} \\ X_{21} & X_{22} & \cdots & X_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ X_{J1} & X_{J2} & \cdots & X_{JS} \end{pmatrix}$$

be the $J \times S$ matrix of payoffs of all securities. X is called the payoff matrix.

Definition 1.2. A portfolio h is the holding of the J securities. Explicitly,

$$h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_J \end{pmatrix} \in \mathbb{R}^J,$$

where h_j is the holding of security j (h_j is a constant, not a random variable).

$$\begin{pmatrix} h_j > 0 & \Longrightarrow \text{buy} \\ h_j < 0 & \Longrightarrow \text{short sale} \end{pmatrix}$$

Definition 1.3. The portfolio payoff

$$\begin{aligned} h^T X &= \begin{pmatrix} h_1 & h_2 & \cdots & h_J \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1S} \\ X_{21} & X_{22} & \cdots & X_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ X_{J1} & X_{J2} & \cdots & X_{JS} \end{pmatrix} \\ &= \left(\sum_{i=1}^{J} h_i X_{i1}, \sum_{i=1}^{J} h_i X_{i2}, \cdots, \sum_{i=1}^{J} h_i X_{iS} \right) \end{aligned}$$

Moreover, $\sum_{i=1}^{J} h_i X_{is}$ means the payoff at the state s at time 1.

Example 1.4. Suppose that there are 2 securities with 3 states. The payoff of the securities is shown in Figure 1.3.

Then the payoff matrix is given by

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 20 & 40 & 60 \\ 120 & 100 & 80 \end{pmatrix}.$$

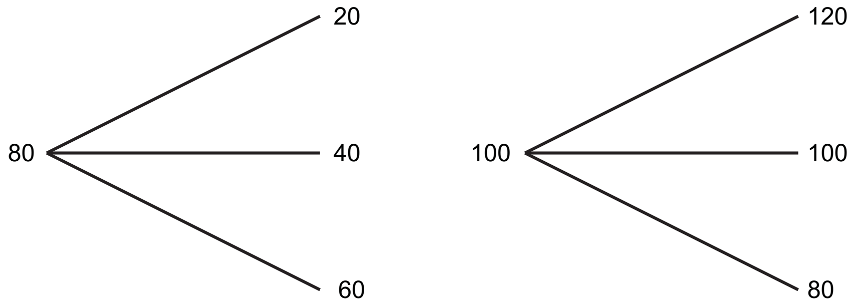

FIGURE 1.3. $X_1 = (20, 40, 60)$, $X_2 = (120, 100, 80)$

Suppose $h = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, then

$$h^T X = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 20 & 40 & 60 \\ 120 & 100 & 80 \end{pmatrix} = \begin{pmatrix} -80 & -20 & 40 \end{pmatrix}.$$

Definition 1.5. (1) The asset span is defined by

$$\begin{aligned} \mathcal{M} &= \{Z \in \mathbb{R}^S : Z = h^T X \text{ for some } h \in \mathbb{R}^J\} = \{h^T X : h \in \mathbb{R}^J\} \\ &= \operatorname{span}\{X_1, X_2, ..., X_J\}. \end{aligned}$$

(The set of payoffs available via trades in security market.)

(2) If $\mathcal{M} = \mathbb{R}^S$, the market is called complete.

(3) If $\mathcal{M} \neq \mathbb{R}^S$, the market is called incomplete.

Example 1.6. (1) As in Example 1.4, the asset span is given by

$$\begin{aligned} \mathcal{M} &= \left\{ \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} 20 & 40 & 60 \\ 120 & 100 & 80 \end{pmatrix} : h_1, h_2 \in \mathbb{R} \right\} \\ &= \{(20h_1 + 120h_2, 40h_1 + 100h_2, 60h_1 + 80h_2) : h_1, h_2 \in \mathbb{R}\} \\ &\neq \mathbb{R}^3. \end{aligned}$$

Thus, the market is incomplete.

(2) Suppose

$$X_1 = (1, 2, 3, 4) \qquad \text{and} \qquad X_2 = (1, 1, 1, 1).$$

Then its payoff matrix

$$X = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and its asset span is given by

$$\begin{aligned}
\mathcal{M} &= \text{span}\{h^T X : h \in \mathbb{R}^2\} \\
&= \left\{ \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} : h_1, h_2 \in \mathbb{R} \right\} \\
&= \{(h_1 + h_2, 2h_1 + h_2, 3h_1 + h_2, 4h_1 + h_2) : h_1, h_2 \in \mathbb{R}\} \\
&= \{h_1(1, 2, 3, 4) + h_2(1, 1, 1, 1) : h_1, h_2 \in \mathbb{R}\}.
\end{aligned}$$

Hence, $\mathcal{M} \neq \mathbb{R}^4 = \mathbb{R}^S$. Thus, the market is incomplete.

(3) Suppose that $J = 4$, $S = 3$ and the payoff at time 1 is given by

$$X_1 = (1, 1, 1), \quad X_2 = (1, 2, 3) \qquad X_3 = (2, 3, 4) \qquad X_4 = (5, 7, 9).$$

Then its payoff matrix is given by

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 7 & 9 \end{pmatrix},$$

and its asset span

$$\begin{aligned}\mathcal{M} &= \text{span}\{X_1, X_2, X_3, X_4\} \\ &= \{h_1(1,1,1) + h_2(1,2,3) + h_3(2,3,4) + h_4(5,7,9) : h_1, h_2, h_3, h_4 \in \mathbb{R}\} \\ &= \{h_1(1,1,1) + h_2(1,2,3) : h_1, h_2 \in \mathbb{R}\} \neq \mathbb{R}^3.\end{aligned}$$

Hence, the market is incomplete.

Theorem 1.7. *A market is complete* $\iff$ $\text{rank}(X) = S$.

Proof. By definition,

$$\begin{aligned}&\text{a market is complete} \\ \iff\ &\mathcal{M} = \mathbb{R}^S. \\ \iff\ &\text{for all } Z \in \mathbb{R}^S \text{ there exists an } h \in \mathbb{R}^J \text{ such that } Z = h^T X. \\ \overset{(*)}{\iff}\ &\text{rank}(X) = S.\end{aligned}$$

$(*)$ can be shown below:

"$\Longleftarrow$" Let $h = X(XX^T)^{-1}Z^T$, then

$$h^T X = (X(XX^T)^{-1}Z^T)^T X = Z(X^T X)^{-1} X^T X = Z.$$

"$\Longrightarrow$" Using the fact $\text{rank}(X) = \dim(\text{range}(X))$. □

Notation 1.8. The price of securities at time 0 are denoted $P = (P_1, P_2, ..., P_J)^T$.

Remark 1.9. The price of portfolio h at security price P is

$$h^T = \sum_{j=1}^{J} h_j P_j.$$

Definition 1.10. The (gross) return on security j is denoted by

$$r_j = \frac{X_j}{P_j}.$$

Example 1.11. Suppose that there are 3 states and 2 securities

security 1: price at time 0 is $P_1 = 0.8$ and payoff at time 1 is $X_1 = (1, 1, 1)$

security 2: price at time 0 is $P_2 = 1.25$ and payoff at time 1 is $X_2 = (1, 2, 2)$

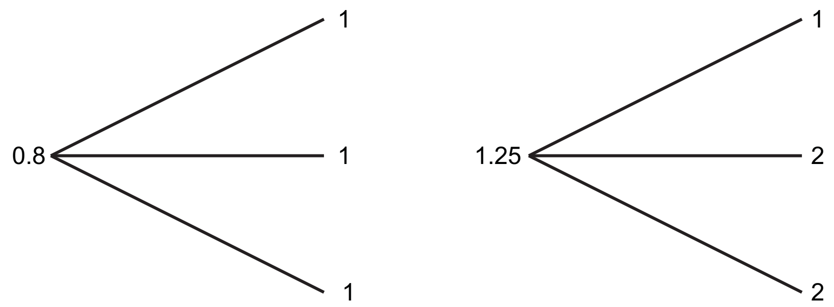

Figure 1.4

$$\text{payoff matrix:} \quad X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$\begin{aligned} \text{asset span } \mathcal{M} &= \left\{ \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} : \ h_1, \ h_2 \in \mathbb{R} \right\} \\ &= \{(h_1 + h_2, \ h_1 + 2h_2, \ h_1 + 2h_2) : h_1, \ h_2 \in \mathbb{R}\} \end{aligned}$$

This implies that dim$\mathcal{M}$=2 (i.e., rank$(X) = 2$). Therefore, the market is not complete.

The security returns are

$$\begin{aligned} r_1 &= \frac{X_1}{P_1} = \frac{(1, \ 1, \ 1)}{0.8} = (1.25, \ 1.25, \ 1.25), \\ r_2 &= \frac{X_2}{P_2} = \frac{(1, \ 2, \ 2)}{1.25} = (0.8, \ 1.6, \ 1.6). \end{aligned}$$

1.2. Arbitrage and strong arbitrage

Notation 1.12. For $x, y \in \mathbb{R}^n$,

$$\begin{aligned} x \geq y &\iff x_i \geq y_i \text{ for all } i, \\ x > y &\iff x \geq y \text{ and } x \neq y, \\ x \gg y &\iff x_i > y_i \text{ for all } i. \end{aligned}$$

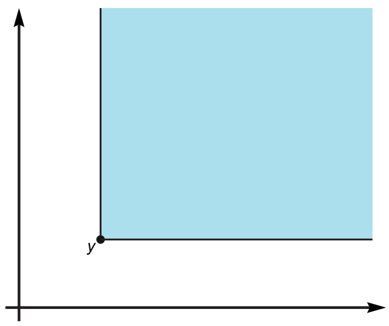

FIGURE 1.5. $\{x : x \geq y\}$

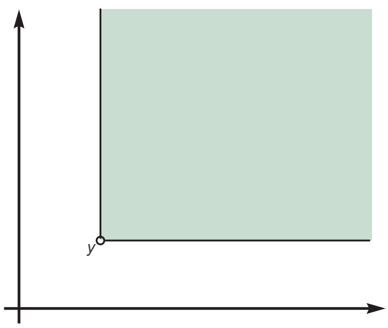

FIGURE 1.6. $\{x : x > y\}$

The sets $\{x : x \geq y\}$, $\{x : x > y\}$, and $\{x : x \gg y\}$ are shown in Figures 1.5, 1.6, and 1.7, respectively.

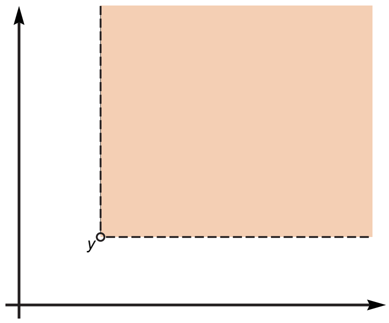

FIGURE 1.7. $\{x : x \gg y\}$

Definition 1.13. (1) A strong arbitrage is a portfolio h such that

$$h^T X \geq 0 \qquad \text{and} \qquad h^T P < 0.$$

(2) An arbitrage is a portfolio h such that

$$h^T X \geq 0 \qquad \text{and} \qquad h^T P \leq 0$$

with at least one strict inequality.[1]

Remark 1.14. Clearly, a strong arbitrage is an arbitrage. But there is a portfolio which can be an arbitrage, but not a strong arbitrage.

Example 1.15. Suppose that there are two securities in the market:

$$\text{security 1 (bond):} \qquad P_1 = 1, \qquad X_1 = (1,1);$$
$$\text{security 2 (stock):} \qquad P_2 = 1, \qquad X_2 = (1,2).$$

[1] Or written as

$$h^T X \geq 0 \qquad \text{and} \qquad h^T P < 0$$

or

$$h^T X > 0 \qquad \text{and} \qquad h^T P \leq 0.$$

Then for $h = (-1,1)^T$,

$$\begin{aligned} h^T P &= (-1,1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -1+1 = 0. \\ h^T X &= \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \geq 0 \end{aligned}$$

Thus, h is an arbitrage, but not a strong arbitrage.

Definition 1.16. (1) If there exists an arbitrage in a market model, we say that there exists an arbitrage opportunity in security market.

(2) If there is no arbitrage opportunity in a market model, we say that there is no arbitrage (arbitrage-free, or no free-lunch) in the model.

(3) Similar definition for strong arbitrage opportunity, no strong arbitrage.

Lemma 1.17. *If there does not exist h such that $h^T X > 0$, then there is no arbitrage, or if we can show that*

$$h^T X \geq 0 \quad \Longrightarrow \quad h = 0,$$

then there does not exist a strategy h such that $h^T X > 0$.

Proof. Exercise. □

Remark 1.18. No arbitrage implies no strong arbitrage.

Example 1.19. (1) Consider a model with two securities and two states.

security 1 (bond): $P_1 = 1,\quad X_1 = (1,1);$

security 2 (stock): $P_2 = 1,\quad X_2 = (1,2).$

Then for $h = (1, -1)^T$,

$$\begin{aligned} h^T P &= (-1, 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0, \\ h^T X &= (-1, 1) \begin{pmatrix} 1/2 & 2 \\ 1/3 & 3/2 \end{pmatrix} = \left(\frac{1}{6}, \frac{1}{2} \right) \gg 0. \end{aligned}$$

Hence, h is a strong arbitrage.

(2) Suppose that the securities have the payoffs

$$X_1 = (-1, 2, 0), \qquad X_2 = (2, 2, -1).$$

A portfolio $h = (h_1, h_2)^T$ has a positive payoff

$$h^T X = \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 2 & 2 & -1 \end{pmatrix} \geq 0,$$

i.e.,

$$(-h_1 + 2h_2,\ 2h_1 + 2h_2,\ -h_2) \geq 0.$$

Thus, we have to solve the system of linear equations

$$\begin{cases} -h_1 + 2h_2 \geq 0 \\ 2h_1 + 2h_2 \geq 0 \\ -h_2 \geq 0 \end{cases}$$

Hence, we get that

$$h_2 \geq \frac{1}{2} h_1 \tag{1.1}$$

$$h_1 + h_2 \geq 0 \tag{1.2}$$

$$h_2 \leq 0 \tag{1.3}$$

Due to (1.1) and (1.3), we have

$$h_1 \leq 0, \ \ h_2 \leq 0. \tag{1.4}$$

Hence,

$$h_1 + h_2 \leq 0.$$

Together with (1.2), we get

$$h_1 + h_2 = 0.$$

By (1.4) this results

$$h_1 = h_2 = 0,$$

i.e,,

$$h = 0.$$

Thus, there is no arbitrage in this model.

1.3. Positivity and the payoff pricing functional

1.3.1. The payoff pricing functional.

Definition 1.20 (The law of one price). All portfolios with the same payoff have the same price, i.e., if h, k are portfolios, then

$$h^T X = k^T X \qquad \Longrightarrow \qquad h^T P = k^T P.$$

Proposition 1.21. *No strong arbitrage* $\Longrightarrow$ *the law of one price holds.*

Proof. Suppose the law of one price does not hold. Then there exist two portfolios h and k such that

$$h^T X = k^T X, \qquad \text{but} \qquad h^T P > k^T P.$$

Let $l = k - h$, then

$$\begin{aligned} l^T X &= k^T X - h^T X = 0, \\ l^T P &= k^T P - h^T P < 0. \end{aligned}$$

This implies that l is a strong arbitrage. □

Remark 1.22. The converse of Proposition 1.21 does not hold. For example, consider a model with two securities and two states. Suppose that the price at time 0: $P = (P_1, P_2)^T = (1/4, 1)^T$ and the payoff at time 1: $X_1 = (1, 1)$, $X_2 = (2, 3)$, i.e., the payoff matrix is given by

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}.$$

Assume that $h^T X = k^T X$. Since X is invertible, $h = k$. Thus,

$$h^T P = k^T P.$$

This implies that the law of one price holds. However, $h = (4, -1)^T$ is a strong arbitrage, since

$$\begin{aligned} h^T P &= (4, -1) \begin{pmatrix} 1/4 \\ 1 \end{pmatrix} = 0, \\ h^T X &= (4, -1) \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = (2, 1) \gg 0. \end{aligned}$$

Remark 1.23. The law of one price holds $\iff$ Every portfolio with zero payoff has zero price.

Definition 1.24. Define $q : \mathcal{M} \to \mathbb{R}$ that assigns to each payoff the price of the portfolio that generates the payoff, i.e.,

$$q(Z) = \{w : w = h^T P \text{ for some } h \text{ such that } Z = h^T X\}.$$

If the law of one price holds, we call q the payoff pricing functional (i.e., q is well-defined).

Example 1.25. (1) Consider a financial market with 3 securities and 3 states. Let $P = (P_1, P_2, P_3)^T = (1, 1, 1)^T$ and the payoff matrix

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1/2 \\ 1/2 & 1 & 2 \end{pmatrix}.$$

For $Z = (9, 7, 9)$, since $Z = h^T X$, then the corrspondng portfolio h satisifes

$$(9, 7, 9) = (h_1, h_2, h_3) \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1/2 \\ 1/2 & 1 & 2 \end{pmatrix}.$$

This imples that

$$\begin{aligned} (h_1, h_2, h_3) &= (9, 7, 9) \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1/2 \\ 1/2 & 1 & 2 \end{pmatrix}^{-1} \\ &= (9, 7, 9)(-\frac{4}{3}) \begin{pmatrix} 3/2 & -1 & -1/2 \\ -15/4 & 3/2 & 3/2 \\ 3/2 & -1/2 & -1 \end{pmatrix} \\ &= (-1, 4, 4). \end{aligned}$$

Hence, the corresponding portfolio $h = \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}$. Thus,

$$q(9,7,9) = h^T P = (-1,4,4) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 7.$$

More general, for $Z = (Z_1, Z_2, Z_3)$, the corresponding portfolio

$$h = (Z_1, Z_2, Z_3)X^{-1} = \left(-2Z_1 + 5Z_2 - 2Z_3, \frac{4}{3}\,Z_1 - 2Z_2 + \frac{2}{3}\,Z_3, \frac{2}{3}\,Z_1 - 2Z_2 + \frac{4}{3}\,Z_3\right).$$

Thus, the payoff pricing functional is of the form

$$\begin{aligned}
q(Z) &= q(h^T X) = h^T P \\
&= \left(-2Z_1 + 5Z_2 - 2Z_3, \frac{4}{3}\,Z_1 - 2Z_2 + \frac{2}{3}\,Z_3, \frac{2}{3}\,Z_1 - 2Z_2 + \frac{4}{3}\,Z_3\right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= (-2Z_1 + 5Z_2 - 2Z_3) + \left(\frac{4}{3}\,Z_1 - 2Z_2 + \frac{2}{3}\,Z_3\right) + \left(\frac{2}{3}\,Z_1 - 2Z_2 + \frac{4}{3}\,Z_3\right) \\
&= Z_2.
\end{aligned}$$

(2) Consider a financial market with 2 securities and 3 states.

$$\text{price} \qquad P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$$

$$\text{payoff matrix} \qquad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

Then the asset span

$$\begin{aligned}\mathcal{M} &= \{h_1(1,2,1) + h_2(2,2,2) : h_1, h_2 \in \mathbb{R}\} \\ &= \{(h_1 + 2h_2, 2h_1 + 2h_2, h_1 + 2h_2) : h_1, h_2 \in \mathbb{R}\} \\ &= \{(x, y, x) : x, y \in \mathbb{R}\}.\end{aligned}$$

Hence,

$$\begin{aligned}q((x,y,x)) &= q\left((y-x)(1,2,1) + \left(x - \frac{y}{2}\right)(2,2,2)\right) \\ &= (y-x)\cdot 1 + \left(x - \frac{y}{2}\right)\cdot \frac{3}{2} = \frac{x}{2} + \frac{y}{4}.\end{aligned}$$

(3) Consider a financial market with 2 securities and 3 states. Let

$$\begin{aligned}P &= \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ X &= \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}.\end{aligned}$$

Then the asset span

$$\begin{aligned}\mathcal{M} &= \{h_1(1,1,1) + h_2(2,3,4) : h_1, h_2 \in \mathbb{R}\} \\ &= \{(h_1 + 2h_2, h_1 + 3h_2, h_1 + 4h_2) : h_1, h_2 \in \mathbb{R}\} \\ &= \{(x, y, -x + 2y) : x, y \in \mathbb{R}\}.\end{aligned}$$

Hence,

$$\begin{aligned}q((x,y,-x+2y)) &= q\left((3x-2y)(1,1,1) + (-x+y)(2,3,4)\right) \\ &= (3x-2y)\cdot 1 + (-x+y)\cdot 1 = 2x - y.\end{aligned}$$

Theorem 1.26. *q is a payoff pricing functional* $\iff$ *$q : \mathcal{M} \to \mathbb{R}$ is linear.*

Proof. "$\Longrightarrow$" If q is a payoff pricing functional, then the law of one price holds, i.e.,

$$\text{“}h^T X = k^T X \quad \Longrightarrow \quad h^T P = k^T P\text{”}. \tag{1.5}$$

Consider $q(cZ_1 + Z_2) = w$. Then there exists a portfolio h such that $w = h^T P$ and $cZ_1 + Z_2 = h^T X$.

Let $q(Z_1) = w_1,\ q(Z_2) = w_2$, then

$$\begin{cases} \text{there exists } h_1 \text{ such that } w_1 = h_1^T P \text{ and } Z_1 = h_1^T X, \\ \text{there exists } h_2 \text{ such that } w_2 = h_2^T P \text{ and } Z_2 = h_2^T X. \end{cases}$$

Hence,

$$h^T X = cZ_1 + Z_2 = ch_1^T X + h_2^T X = (ch_1 + h_2)^T X.$$

By (1.5), we have

$$\begin{aligned} & w = h^T P = (ch_1 + h_2)^T P = ch_1^T P + h_2^T P = cw_1 + w_2 \\ \Longrightarrow\quad & q(cZ_1 + Z_2) = w = cw_1 + w_2 = cq(Z_1) + q(Z_2) \\ \Longrightarrow\quad & q \text{ is linear.} \end{aligned}$$

"$\Longleftarrow$" Suppose that $q : \mathcal{M} \longrightarrow \mathbb{R}$ is linear, then q is a functional. This implies that q is well-defined. Hence, the law of one price holds. $\square$

1.3.2. Positivity of functional.

Definition 1.27. Let $\mathcal{L}$ be a vector subspace of $\mathbb{R}^n$ and let $F : \mathcal{L} \longrightarrow \mathbb{R}$ is a functional.

(1) F is said to be positive on $\mathcal{L}$ if $F(x) \geq 0$, for all $x \in \mathcal{L}$ with $x \geq 0$.
(2) F is strictly positive on $\mathcal{L}$ if $F(x) > 0$, for all $x \in \mathcal{L}$ with $x > 0$.

Remark 1.28. If F is a (strictly) positive functional on $\mathbb{R}^n$, we say that F is a (strictly) positive functional.

Example 1.29. (1) Define $F(x_1, x_2) = x_2$. Then F is positive, but F is not strictly positive (since $F(0,1) = 0$).
(2) The function $F(x_1, x_2) = x_1 + x_2$ is strictly positive.

Remark 1.30. If $\mathcal{L} = \mathcal{R}^n$, i.e., F is a linear functional, then there exists a vector $a \in \mathbb{R}^n$ such that $F(x) = a \cdot x$ for $x \in \mathbb{R}^n$, i.e.,

$$F(x_1, x_2, ..., x_n) = a_1 x_1 + a_2 x_2 + ... + a_n x_n.$$

Remark 1.31. If $F(x) = a \cdot x$ for $a \in \mathbb{R}^n$. Then

(1) F is positive $\iff$ $a \geq 0$.
(2) F is strictly positive $\iff$ $a \gg 0$.

But this result does not hold if the domain of F is not $\mathbb{R}^n$, but its proper subspace. For example, let

$$\mathcal{L} = \{(x, y, -x-y) : x, y \in \mathbb{R}\},$$

which is a proper subspace of $\mathbb{R}^3$, the function

$$F(x, y, -x-y) = -x + y = (-1, 1) \cdot (x, y)$$

is a strictly positive functional on $\mathcal{L}$, but is not of the form given above.

Theorem 1.32. *Suppose the market is complete, i.e., $\mathcal{M} = \mathbb{R}^S$.*

(1) *The payoff pricing functional q is linear and strictly positive* $\iff$ *There is "no arbitrage".*

(2) *The payoff pricing functional is linear and positive.* $\iff$ *There is no strong arbitrage.*

PROOF. (1) "$\Longrightarrow$" Suppose that q is linear and strictly positive. Then there exists $a \in \mathbb{R}^S$ such that $q(Z) = a \cdot Z$ for some $a \gg 0$.

Claim that the model is arbitrage-free, i.e., there exists a portfoloe h such that

$$\text{" if } h^T X > 0 \quad \Longrightarrow \quad h^T P \leq 0\text{"} \qquad \text{and} \qquad \text{"" if } h^T X = 0 \quad \Longrightarrow \quad h^T P = 0\text{".}$$

(i) Suppose that $Z = h^T X > 0$. Since q is strictly positive,

$$h^T P = q(h^T X) = q(Z) > 0.$$

(ii) Suppose that $Z = h^T X = 0$. Since q is linear,

$$h^T P = q(h^T X) = q(Z) = 0.$$

By (i) and (ii) we know that there is no arbitrage in this market.

"$\Longleftarrow$" Suppose there is no arbitrage.

(i) Claim that q is linear.

Assume that the law of one price fails, i.e., there exist portfolios h and $k \in \mathbb{R}^J$ such that $h^T X = k^T X$ but $h^T P \neq k^T P$. Without loss of generality, we may assume $h^T P < k^T P$. Consider a portfolio $m = h - k$, then

$$\begin{aligned} m^T X &= (h-k)^T X = h^T X - k^T X = 0 \\ m^T P &= (h-k)^T P = h^T P - k^T P < 0. \end{aligned}$$

Thus, m is an arbitrage which leads to a contradiction. This implies that the law of one price holds. By Theorem 1.26 we know that q is linear.

(ii) Claim that q is strictly positive.

For $Z \in \mathcal{M}$, there exists $h \in \mathbb{R}^J$ such that $q(z) = h^T P$ with $Z = h^T X$. Due to the assumption of "no arbitrage", we have

$$\text{if } Z = h^T X > 0 \qquad \Longrightarrow \qquad q(Z) = h^T P > 0$$

and

$$\text{if } Z = h^T X = 0 \qquad \Longrightarrow \qquad q(Z) = h^T P = 0.$$

Thus, q is strictly positive.

(2) Exercise

□

Corollary 1.33. *Suppose that the market is incomplete, i.e., $\mathcal{M} \subsetneqq \mathbb{R}^S$. Then the results of Theorem 1.32 still holds, i.e.,*

(1) *The payoff pricing functional q is linear and strictly positive on $\mathcal{M}$ $\Longleftrightarrow$ There is "no arbitrage" in the model.*
(2) *The payoff pricing functional is linear and positive on $\mathcal{M}$ $\Longleftrightarrow$ There is no strong arbitrage in the model.*

Remark 1.34. Suppose the model is not complete, then without loss of generality, we assume that

$$\begin{aligned} \mathcal{M} &= \{(y_1, ..., y_n, g_{n+1}(y_1, ..., y_n), g_S(y_1, ..., y_n)) : y_1, ..., y_n \in \mathbb{R}\} \\ &= \{(y, g_{n+1}(y), ..., g_S(y)) : y \in \mathbb{R}^n\} \subsetneqq \mathbb{R}^S, \end{aligned} \tag{1.6}$$

where $g_{n+1}, ..., g_S$ are linear functional of $y_1, ...y_n$. Due to Remark 1.31 we know that a positive linear functional on $\mathcal{M}$ is not necessary to be of the form $a \cdot x$ for all $x \in \mathcal{M}$ with

$a \geq 0$. However, a linear functional $F : \mathcal{M} \longrightarrow \mathbb{R}$ of the form

$$F((y, g_{n+1}(y), ..., g_S(y))) = a \cdot y \qquad \text{for all } y \in \mathbb{R}^n$$

with a positive vector $a \in \mathbb{R}^n$ does imply that F is a positive linear functional on $\mathcal{M}$.

Thus, we have the following corollary.

Corollary 1.35. *Suppose the market is incomplete and the asset span $\mathcal{M}$ is of the form (1.6).*

(1) *If there exists a strictly positive vector $a \in \mathbb{R}^n$ such that*

$$q((y, g_{n+1}(y), ..., g_S(y))) = a \cdot y \qquad \textit{for all } y \in \mathbb{R}^n,$$

then there is no arbitrage in the model.

(2) *If there exists a positive vector $a \in \mathbb{R}^n$ such that*

$$q((y, g_{n+1}(y), ..., g_S(y))) = a \cdot y \qquad \textit{for all } y \in \mathbb{R}^n,$$

then there is no strong arbitrage in the model.

Remark 1.36. The inverse direction of Corollary 1.35 does not hold. For example, consider a model with two securities and three states

$$P = (P_1, P_2) = (1, 1), \qquad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Then its asset span is given by

$$\mathcal{M} = \text{span}\{(2, 2, 2), (1, 2, 3)\} = \{(x, y, -x + 2y) : x, y \in \mathbb{R}\}$$

and the corresponding pricing valuation functional $q : \mathcal{M} \longrightarrow \mathbb{R}$ is of the form

$$\begin{aligned} q((x, y, -x + 2y)) &= q\left(\frac{1}{2}(2x - y)(2, 2, 2) + (y - x)(1, 2, 3)\right) \\ &= \frac{1}{2}(2x - y) \cdot 1 + (y - x) \cdot 1 = \frac{1}{2}y = \left(0, \frac{1}{2}\right) \cdot (x, y), \end{aligned}$$

i.e., $a = (0, 1/2) \geq 0$. Thus, there is no strong arbitrage in the model. However, from Corollary 1.35 we cannot conclude that the model is arbitrage-free or not. Using Corollary 1.33 (1) we know that if we can show that $q(x, y, -x + 2y) > 0$ for all $(x, y, -x + 2y) > 0$, we can get the model is no arbitrage. $(x, y, -x + 2y) > 0$ means that $x \geq 0$, $y \geq 0$, $-x + 2y \geq 0$ ad with at least one strict inequality. However, if $y = 0$,

$$0 \leq x \leq 2y = 0$$

which leads to a contradiction. Thus, $y > 0$. Moreover, by $q((x, y, -x + 2y)) = \dfrac{y}{2}$ we can see that

$$q((x, y, -x + 2y)) > 0 \qquad \text{if } (x, y, -x + 2y) > 0.$$

In fact, in the next chapter, we shall introduce anther method to show that this model is arbitrage-free.

Example 1.37. (1) As in Example 1.25 (1)

$$q((Z_1, Z_2, Z_3)) = Z_2 = (0, 1, 0) \cdot (Z_1, Z_2, Z_3),$$

i.e., $a = (0, 1, 0) \geq 0$. Since the market is complete, by Theorem 1.32 there is no strong arbitrage, but there exists an arbitrage opportunity in the model. In fact, we can find the portfolio $h = (-2, 1, 1)^T$ as an arbitrage opportunity.

(2) As in Example 1.25 (2)

$$q((x, y, x)) = \frac{x}{2} + \frac{y}{4} = \left(\frac{1}{2}, \frac{1}{4}\right) \cdot (x, y),$$

i.e., $a = \left(\frac{1}{2}, \frac{1}{4}\right) \gg 0$. Thus, there is no arbitrage.

(3) As in Example 1.25 (3)

$$q((x, y, -x + 2y)) = 2x - y = (2, -1) \cdot (x, y),$$

i.e., $a = (2, -1)$. Indeed, there is a strong arbitrage opportunity (and an arbitrage opportunity). However, from the fact $a = (2, -1)$ we cannot conclude that this model is not arbitrage-free.

(4) Consider a market model with two securities and three states. Let $P = (P_1, P_2)^T = (9, 6)^T$, and the payoff matrix is of the form

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 2 & -1 \end{pmatrix}.$$

Then its asset span is given by

$$\mathcal{M} = \operatorname{span}\{(-1, 2, 0), (2, 2, -1)\} = \left\{ \left(x, y, -\frac{1}{3}\,x - \frac{1}{6}\,y\right) : x, y \in \mathbb{R} \right\}$$

and its pricing payoff functional is of the form

$$\begin{aligned} q\left(\left(x, y, -\frac{1}{3}\,x - \frac{1}{6}\,y\right)\right) &= q\left(\frac{1}{3}\,(y - x)(-1, 2 - 0) + \left(\frac{1}{3}\,x + \frac{1}{6}\,y\right)(2, 2, -1)\right) \\ &= \frac{1}{3}\,(y - x)q(-1, 2 - 0) + \left(\frac{1}{3}\,x + \frac{1}{6}\,y\right) q(2, 2, -1) \\ &= \frac{1}{3}\,(y - x) \cdot 9 + \left(\frac{1}{3}\,x + \frac{1}{6}\,y\right) \cdot 6 = -x + 4y \\ &= (-1, 4) \cdot (x, y). \end{aligned}$$

Thus, the corresponding $a = (-1, 4)$. However, we have shown in Example 1.19 that this model is arbitrage-free.

Remark 1.38. Suppose q is a payoff pricing functional, then $q(X_j) = P_j$ for all j.

Proof. Let

$$q(X_j) = \{w : \ w = h^T P \text{ for some } h \text{ such that } X_j = h^T X\}.$$

We aim to choose a portfolio $h \in \mathbb{R}^J$ satisfying $X_j = h^T X$.

Since $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)^T$ satisfies

$$e_j^T X = (0, \cdots, 0, 1, 0, \cdots, 0) \begin{pmatrix} X_1 \\ \vdots \\ X_J \end{pmatrix} = X_j,$$

we may take $h = e_j$. This implies $h^T X = e_j^T X$. By the law of one price, we have $h^T P = e_j^T P = P_j$. This means

$$q(X_j) = P_j \qquad \text{for all } 1 \leq j \leq J.$$

$\square$

1.4. Exercise

(1) Find the payoff matrix, asset span and security returns of the following market model. Furthermore, determine if the market is complete.

  (a) $P = (P_1, P_2)^T = (1, 1)^T$; $X_1 = (1.2, 1.2, 1.2)$, $X_2 = (1, 1.2, 2)$.

  (b) $P = (P_1, P_2, P_3)^T = (1, 1, 1)^T$; $X_1 = (1.2, 1.2, 1.2)$, $X_2 = (0.8, 1, 1.2)$, $X_3 = (0.9, 1.1, 1.3)$.

  (c) $P = (P_1, P_2, P_3, P_4)^T = (1, 1, 1, 1)^T$; $X_1 = (1, 1, 1)$, $X_2 = (0.8, 1.2, 1.2)$, $X_3 = (0.8, 0.8, 1.2)$, $X_4 = (0.5, 2, 4)$.

(2) Determine if there exists arbitrage opportunity in the following model. If yes, find an arbitrage.

  (a) $P = (P_1, P_2)^T = (1, 1)^T$; $X_1 = (1, 1)$, $X_2 = (0.8, 2)$.

(b) $P = (P_1, P_2)^T = (1, 1)^T$; $X_1 = (1.2, 1.2, 1.2)$, $X_2 = (1.1, 2, 2)$.

(c) $P = (P_1, P_2)^T = (1, 1)^T$; $X_1 = (1.2, 1.2, 1.2)$, $X_2 = (1.2, 2, 2)$.

(d) $X_1 = (1, -1, 1, -1)$, $X_2 = (-1, 1, 1, -1)$.

(3) Determine if the law of one price holds. If yes, find the corresponding payoff pricing function.

(a) $P = (P_1, P_2)^T = (1, 1)^T$; $X_1 = (1.2, 1.2, 1.2)$, $X_2 = (0.8, 1, 1.2)$.

(b) $P = (P_1, P_2, P_3)^T = (1, 1, 1)^T$; $X_1 = (1, 1, 1)$, $X_2 = (0.8, 1, 1.2)$, $X_3 = (1.2, 1, 0.8)$.

(4) Determine if the following functionals are positive or strictly positive.

(a) $F(x_1, x_2) = x_1^2 + x_2^3$;

(b) $F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$;

(c) $F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3$;

(d) $F(x_1, x_2, x_3) = 2x_1 + x_2 + 4x_3$;

(e) $F(x_1, x_2, x_3) = 2x_1 - x_2 + 4x_3$;

(f) $F(x_1, x_2, x_3) = 2x_1 + x_3^2$.

(5) Find the payoff pricing functionals in the following cases. Using these functionals to check if the models are arbitrage-free or strong arbitrage-free.

(a) (12 points) $P = (P_1, P_2)^T = (1, 1)^T$; $X_1 = (2, 2, 2)$, $X_2 = (3, 4, 5)$.

(b) (12 points) $P = (P_1, P_2, P_3)^T = (1, 1, 1)^T$; $X_1 = (1, 1, 1)$, $X_2 = (1, 2, 2)$, $X_3 = (2, 1, 2)$.

(6) Let $h \in \mathbb{R}^J$ and X be a $J \times S$ matrix. Prove that if "$h^T X \geq 0$ implies $h = 0$", then there does not exist a vector $h \in \mathbb{R}^J$ such that $h^T X > 0$.

(7) Prove that the law of one price holds if and only if every portfolio with zero profit has zero price.

(8) Prove that the payoff pricing functional is linear and positive if and only if there is no strong arbitrage.