CHAPTER 3

State Prices and Risk-Neutral Probabilities

3.1. State prices

Let $Q : \mathbb{R}^S \to \mathbb{R}$ be a valuation functional and let $(e_s)_{1\leq s\leq S}$ be the standard ordered basis for $\mathbb{R}^S$, i.e., $e_i = (0, ..., 0, 1, 0, ..., 0)$, where 1 is at the i-th position.

Definition 3.1. Define $q_s = Q(e_s)$ to be the state price of state s. If $\mathcal{M} = \mathbb{R}^S$ (complete market), then

$$q_i = Q(e_i) = q(e_i) = \{h^T P : \ h \text{ satisfies } h^T X = e_i\}.$$

Remark 3.2. (1) If Q is strictly positive, then each state price q_i is strictly positive.

(2) If Q is positive, then each state price q_i is positive.

Example 3.3. (1) As in Example 2.16 (1),

$$Q(y_1, y_2) = \frac{1}{2}y_1 + \frac{1}{4}y_2.$$

Thus, the state prices are given by

$$q_1 = Q(1, 0) = \frac{1}{2}, \qquad q_2 = Q(0, 1) = \frac{1}{4}.$$

(2) As in Example 2.16 (2),

$$Q(x, y, z) = \frac{1}{8}\,x + \frac{1}{4}\,y + \frac{1}{8}z.$$

Thus, the state prices are given by

$$q_1 = Q(1,0,0) = \frac{1}{8}, \qquad q_2 = Q(0,1,0) = \frac{1}{4}, \qquad q_3 = Q(0,0,1) = \frac{1}{8}.$$

Remark 3.4. For $Z \in \mathbb{R}^S$, we have

$$Z = \sum_{s=1}^{S} Z_s e_s.$$

The representation

$$Q(Z) = Q\left(\sum_{i=1}^{S} Z_i e_i\right) = \sum_{i=1}^{S} Z_i Q(e_i) = \sum_{i=1}^{S} Z_i q_i = \bar{q} \cdot Z,$$

with $\bar{q} = (q_1, ..., q_S)$, is called the state-price representation of the valuation functional Q.

Remark 3.5. Due to $Q(Z) = \bar{q} \cdot Z$, we see that

$$P_j = Q(X_j) = \bar{q} \cdot X_j = X_j\, \bar{q}^T.$$

This implies that

$$P = X\bar{q}^T. \tag{3.1}$$

Theorem 3.6. *There exists a strictly positive valuation functional* $\Longleftrightarrow$ *there exists a strictly positive solution* $\bar{q}$ *to the equation (3.1). Each strictly positive solution* q *defines a strictly positive valuation functional* Q *such that* $Q(Z) = q \cdot Z$*, for all* $Z \in \mathbb{R}^S$*.*

Proof. "$\Longrightarrow$ Shown above! ($\bar{q} = (q_1, ..., q_S)$)

"$\Longleftarrow$" If $\bar{q}$ is a strictly positive solution to (3.1). Define

$$Q(Z) = \bar{q} \cdot Z.$$

Then Q is linear and strictly positive.

It remains to show that

$$Q(Z) = \text{ the payoff pricing function } q(Z) \text{ for all } Z \in \mathcal{M}.$$

For $Z \in \mathcal{M}$, there exists a portfolio h such that $Z = h^T X$, then

$$Q(Z) = \bar{q} \cdot Z = Z\bar{q}^T = h^T X \bar{q}^T = h^T P = q(Z).$$

$\square$

Corollary 3.7. *No arbitrage* $\iff$ *there exists a vector* $\bar{q} \in \mathbb{R}^S$ *such that*

$$P = X\bar{q}^T \qquad \text{and} \qquad \bar{q} \gg 0.$$

PROOF. Exercise. $\square$

Theorem 3.8. *The following statements are equivalent:*

(1) *There exists a positive valuation functional.*

(2) *No strong arbitrage.*

(3) *There exists a vector* $\bar{q} \in \mathbb{R}^S$ *such that*

$$P = X\bar{q}^T \qquad \text{and} \qquad \bar{q} \geq 0.$$

PROOF. Exercise. $\square$

Example 3.9. (1) There are two securities in the market.

security 1 (bond): $P_1 = 1$, $X_1 = (1, 1)$;

security 2 (stock): $P_2 = 1$, $X_2 = (1, 2)$.

Solve the equation

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

Thus,

$$\bar{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This is no strong arbitrage, but there is an arbitrage opportunity.

(2) There are two securities with 3 states in the market.

security 1 (bond): $P_1 = 1/2, \quad X_1 = (1, 1, 1);$

security 2 (stock): $P_2 = 1, \quad X_2 = (1, 2, 4).$

Positive state prices q_1, q_2, q_3 are positive solutions to

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}.$$

Thus,

$$\begin{cases} q_1 + q_2 + q_3 &= \dfrac{1}{2}, \\ q_1 + 2q_2 + 4q_3 &= 1, \end{cases}$$

whose solution is

$$q_1 = 2q_3, \qquad q_2 = \frac{1}{2} - 3q_3.$$

(i) For positive state prices: $q_1, q_2, q_3 \geq 0 \quad \Longrightarrow \quad 0 \leq q_3 \leq \dfrac{1}{6}.$

Thus, there exists a positive solution to Equation (3.1), i.e., this model is strong arbitrage free.

(ii) For state prices: strictly positive $\Longrightarrow$ $q_1, q_2, q_3 > 0$ $\Longrightarrow$ $0 < q_3 < \frac{1}{6}$.

Thus, there exists a strictly positive solution to Equation (3.1), i.e., this model is no arbitrage.

(3) There are two securities in the market.

$$\begin{array}{lll} \text{security 1 (bond):} & P_1 = 1, & X_1 = (1, 1, 1); \\ \text{security 2 (stock):} & P_2 = 1, & X_2 = (1, 2, 4). \end{array}$$

Positive state prices q_1, q_2, q_3 are positive solutions to

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}.$$

Thus,

$$\begin{cases} q_1 + \ \ q_2 + \ \ q_3 &= 1, \\ q_1 + 2q_2 + 4q_3 &= 1, \end{cases}$$

whose solution is

$$q_1 = 1 + 2q_3, \qquad q_2 = -3q_3.$$

Thus,

(i) Since this system of equations has no strictly positive solution (there is no q_2 and q_3 such that $q_2, q_3 > 0$), this means that this model has an arbitrage opportunity.

(ii) There is one positive solution $\bar{q} = (q_1, q_2, q_3) = (1, 0, 0)$, hence there is no strong arbitrage in this model.

3.2. State prices and value bounds

Proposition 3.10. *No arbitrage* $\Longrightarrow$

$$
\begin{aligned}
q_u(Z) &= \max_{\bar q \geq 0}\{\bar q \cdot Z : P = X\bar q^T\},\\
q_l(Z) &= \min_{\bar q \geq 0}\{\bar q \cdot Z : P = X\bar q^T\}.
\end{aligned}
$$

Proof. Step 1: Let

$$\hat q_u(Z) = \max_{\bar q \geq 0}\{\bar q \cdot Z : P = X\bar q^T\}.$$

For $h \in \mathbb{R}^J$ satisfying $h^T X \geq Z$, we have

$$h^T X \bar q^T \geq Z \bar q^T \qquad \text{for } \bar q \text{ satisfying } P = X\bar q^T \text{ and } \bar q \geq 0.$$

Thus,

$$h^T P \geq \bar q \cdot Z$$

for all trading strategy h satisfying $h^T X \geq Z$. Taking minimum on the left-hand side over all h satisfying $h^T X \geq Z$, we get that $q_u(Z) \geq \bar q \cdot Z$ for all $\bar q$ satisfying $P = X\bar q^T$ and $\bar q \geq 0$. Hence, $q_u(Z) \geq \hat q_u(Z)$.

Step 2: Due to the proof of Theorem 2.14, there exists a valuation functional Q such that $Q(Z) = q_u(Z)$. Thus, $q_u(Z)$ is the least upper bound for the valuation functional. Hence, $q_u(Z) \leq \hat q_u(Z)$ due to the definition of $\hat q_u(Z)$.

By Step 1 and Step 2, we have $q_u(Z) = \hat q_u(Z)$. Similar argument for $q_l(Z)$. $\square$

Example 3.11. (1) There is one security with 2 states in the market. Security $P_1 = 1$, $X_1 = (1, 2)$. Consider the contingent claim $Z = (1, 1)$.

$$
\begin{aligned}
q_u(1,1) &= \max_{\bar q \geq 0}\{\bar q \cdot (1,1) : P = X\bar q^T\}\\
&= \max_{q_1, q_2 \geq 0}\{q_1 + q_2 : q_1 + 2q_2 = 1\}.
\end{aligned}
$$

It is easy to get that $q_u(1,1) = 1$ (See Figure 3.1).

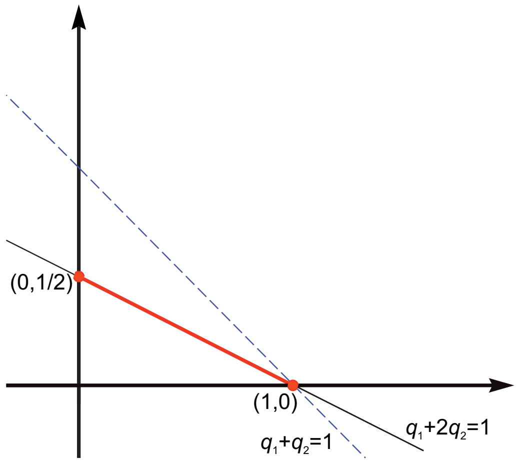

FIGURE 3.1

Similarly, we have

$$q_l(1,1) = \min_{q_1,q_2 \geq 0} \{q_1 + q_2 : q_1 + 2q_2 = 1\} = \frac{1}{2}.$$

(2) There is one security with 3 states in the market with

$$P_1 = 1 \qquad X_1 = (1,2,2).$$

For contingent claim $Z = (1,1,1)$, find $q_u(Z)$ and $q_l(Z)$.

$$\begin{aligned} q_u(1,1,1) &= \max_{\bar{q} \geq 0} \{\bar{q} \cdot (1,1,1) : P = X\bar{q}^T\} \\ &= \max_{q_1,q_2,q_3 \geq 0} \{q_1 + q_2 + q_3 : q_1 + 2q_2 + 2q_3 = 1\}. \end{aligned}$$

Thus, we have

$$q_1 = 1 - 2q_2 - 2q_3 \geq 0, \qquad q_2 \geq 0, \qquad q_3 \geq 0.$$

This implies that we aim to solve

$$\max(1 - q_2 - q_3)$$

subject to the constraints

$$q_2 + q_3 \le \frac{1}{2}, \qquad q_2 \ge 0, \qquad q_3 \ge 0.$$

This implies that $q_u(Z) = 1$. Similarly, we have

$$q_l(Z) = \min_{q_1,q_2,q_3 \ge 0} \{q_1 + q_2 + q_3 : q_1 + 2q_2 + 2q_3 = 1\} = \frac{1}{2}.$$

(3) There are two securities with 3 states in the market.

$$\text{Security 1:} \qquad P_1 = \frac{1}{2}, \qquad X_1 = (1,1,1),$$
$$\text{Security 2:} \qquad P_2 = 1, \qquad X_2 = (1,2,4).$$

For $Z = (0,0,1)$, find $q_u(Z)$ and $q_l(Z)$.

$$\begin{aligned} q_u(Z) &= \max_{\bar{q} \ge 0} \{\bar{q} \cdot Z : P = X\bar{q}^T\} \\ &= \max_{q_1,q_2,q_3 \ge 0} \left\{ q_3 : q_1 + q_2 + q_3 = \frac{1}{2}, q_1 + 2q_2 + 4q_3 = 1 \right\}. \end{aligned}$$

Solving the system of equations

$$\begin{cases} q_1 + \ \ q_2 + \ \ q_3 &= \dfrac{1}{2}, \\ q_1 + 2q_2 + 4q_3 &= 1, \end{cases}$$

we get

$$q_1 = 2q_3 \ge 0 \qquad \text{and} \qquad q_2 = \frac{1}{2} - 3q_3.$$

Thus, $0 \le q_3 \le \frac{1}{6}$. Hence

$$q_u(Z) = \frac{1}{6} \qquad \text{and the corresponding } \bar{q} = \left(\frac{1}{3}, 0, \frac{1}{6}\right)$$

$$q_l(Z) = 0 \qquad \text{and the corresponding } \bar{q} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

3.3. Risk-neutral probabilities

Assumption. There is no arbitrage in the market.

Definition 3.12. A contingent claim that does not depend on the state is risk-free. We denote the risk-free return by $\bar{r}(\in \mathbb{R})$.

Example 3.13. A model with two securities:

$$\text{Security 1:} \qquad P_1 = 1, \qquad X_1 = (2,2,2),$$

$$\text{Security 2:} \qquad P_2 = 1, \qquad X_2 = (1,2,4).$$

Then its risk-free return $\bar{r} = 2$.

Remark 3.14. No arbitrage $\quad\Longrightarrow\quad$ the risk-free return $\bar{r}$ is unique.

Lemma 3.15. *If there is no arbitrage and a nonzero risk-free payoff in the asset span* $\mathcal{M}$*, then*

$$\bar{r} = \frac{1}{\sum_{s=1}^{S} q_s}.$$

Proof. Since the risk-free return $= \bar{r}$, if the price if risk-free asset at time 0 is P, the payoff at time 1 is equal to $(P\bar{r}, ..., P\bar{r})$. Hence the valuation functional

$$\begin{aligned} P &= Q(P\bar{r}, ..., P\bar{r}) = Q\left(\sum_{s=1}^{S} P\bar{r}e_s\right) = \sum_{s=1}^{S} P\bar{r}Q(e_s) \\ &= \sum_{s=1}^{S} P\bar{r}q_s = P\bar{r}\sum_{s=1}^{S} q_s. \end{aligned}$$

Thus,

$$\bar{r} = \frac{1}{\sum_{s=1}^{S} q_s}.$$

$\square$

Remark 3.16. (1) Note that Lemma 3.15 holds if $(1, 1, ..., 1)$ in the asset span $\mathcal{M}$, otherwise, the risk-free return may not be unique.

(2) Suppose $Q(Z) = \bar{q} \cdot Z$, then

$$Q(1, 1, ..., 1) = \sum_{s=1}^{S} q_s = \frac{1}{\bar{r}}.$$

Example 3.17. (1) A model with two securities

Security 1: $P_1 = 1, \quad X_1 = (1, 2, 3),$

Security 2: $P_2 = 2, \quad X_2 = (5, 4, 3).$

Then $(1, 1, 1) \in \mathcal{M}$. Consider the equation

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},$$

i.e., we aim to solve the system of equations

$$\begin{cases} q_1 + 2q_2 + 3q_3 & = 1, \\ 5q_1 + 4q_2 + 3q_3 & = 2, \end{cases}$$

Summing these two equations, we get

$$q_1 + q_2 + q_3 = \frac{1}{2}.$$

This implies that $\bar{r} = \dfrac{1}{\sum_{s=1}^{S} q_s} = 2.$

(2) A model with two securities

Security 1: $P_1 = 1, \quad X_1 = (1, 2, 3),$

Security 2: $P_2 = 2, \quad X_2 = (8, 4, 2).$

Then $(1,1,1) \notin \mathcal{M}$. Consider the equation

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 8 & 4 & 2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \tag{3.2}$$

i.e., we aim to solve the system of equations

$$\begin{cases} q_1 + 2q_2 + 3q_3 &= 1, \\ 8q_1 + 4q_2 + 2q_3 &= 2. \end{cases}$$

Thus, the collection of all strictly positive solutions to (3.2) is given by

$$\left\{ (q_1, q_2, q_3) : q_1 = \frac{2}{3} q_3, \quad q_2 = \frac{1}{2} - \frac{11}{6} q_3, \quad 0 < q_3 < \frac{3}{11} \right\}.$$

Hence,

$$q_1 + q_2 + q_3 = \frac{1}{2} - \frac{1}{6} q_3,$$

which is not unique. In fact, since

$$\frac{5}{11} < q_1 + q_2 + q_3 < \frac{1}{2},$$

we get that the range of the possible risk-free return is $2 < \bar{r} < \frac{11}{5}$ in an arbitrage-free model.

Definition 3.18. Let $\bar{q}$ be strictly positive state price vector. Define

$$\pi_i^* = \frac{q_i}{\sum_{k=1}^{S} q_k} \qquad \text{for all } 1 \le i \le S.$$

We call $\pi^* = (\pi_1^*, \pi_2^*, ..., \pi_S^*)$ the risk-neutral probability ($\pi_s^* > 0$, $\sum_i \pi_i^* = 1$).

Notation 3.19. Let $\mathbb{E}^*$ denote the expectation with respect to the probability π^*, then

$$\mathbb{E}^*[Z] = \sum_{i=1}^{S} Z_i \pi_i^*$$

for contingent claim $Z = (Z_1, ..., Z_S)$.

Remark 3.20. Since

$$\mathbb{E}^*[Z] = \sum_{i=1}^{S} Z_i \pi_i^* = \sum_{i=1}^{S} Z_i \frac{q_i}{\sum_{k=1}^{S} q_k} = \bar{r} \sum_{i=1}^{S} Z_i q_i = \bar{r}\bar{q} \cdot Z$$

This implies that

$$Q(Z) = \bar{q} \cdot Z = \frac{1}{\bar{r}} \mathbb{E}^*[Z]$$

Thus, the value of each contingent claim = the discounted expectation of the claim with respect to the risk-neutral probabilities. i.e.,

$$Q(Z) = \frac{1}{\bar{r}} \mathbb{E}^*[Z],$$

or

$$P_j = Q(X_j) = \frac{1}{\bar{r}} \mathbb{E}^*[X_j].$$

Proposition 3.21.

(1) *No arbitrage* $\iff$ *there exists a risk-neutral probability and* $\pi^* \gg 0$.

(2) *No strong arbitrage* $\iff$ *there exists a risk-neutral probability with* $\pi^* \geq 0$.

Proof. Exercise. □

Remark 3.22. The risk-neutral probability is unique $\iff$ The market is complete.

Theorem 3.23.

$$\begin{aligned} q_u(Z) &= \frac{1}{\bar{r}} \sup_{\pi^* \gg 0} \mathbb{E}^*[Z], \\ q_l(Z) &= \frac{1}{\bar{r}} \inf_{\pi^* \gg 0} \mathbb{E}^*[Z]. \end{aligned}$$

PROOF. Exercise. □

Example 3.24. There are two securities in the market.

$$\begin{aligned} &\text{Security 1:} \quad P_1 = \frac{1}{2}, \quad X_1 = (1,1,1) \\ &\text{Security 2:} \quad P_2 = 1, \quad X_2 = (1,2,4). \end{aligned}$$

Then the risk-free return $\bar{r} = 2$. Therefore,

$$\begin{cases} \pi_1^* + \pi_2^* + \pi_3^* = 1, \\ \pi_1^* + 2\pi_2^* + 4\pi_3^* = 2. \end{cases}$$

Hence, if the model is arbitrage-free,

$$\pi_1^* = 2\pi_3^*, \quad \pi_2^* = 1 - 3\pi_3^*, \quad \text{for } 0 < \pi_3^* < \frac{1}{3},$$

i.e., the collection of all risk-neutral probabilities is given by

$$\Pi = \left\{ (\pi_1^*, \pi_2^*, \pi_3^*) : \pi_1^* = 2\pi_3^*, \quad \pi_2^* = 1 - 3\pi_3^*, \quad 0 < \pi_3^* < \frac{1}{3} \right\}.$$

Hence, for $Z = (1,3,2)$,

$$\mathbb{E}^*[Z] = 1 \cdot \pi_1^* + 3 \cdot \pi_2^* + 2 \cdot \pi_3^* = 3 - 5\pi_3^*.$$

This implies that

$$\begin{aligned} q_u(Z) &= \frac{1}{\bar{r}} \sup_{\pi^* \gg 0} \mathbb{E}^*[Z] = \frac{1}{2} \sup_{\pi^* \in \Pi} \mathbb{E}^*[Z] = \frac{1}{2} \sup_{0 < \pi_3^* < 1/3} (3 - 5\pi_3^*) = \frac{3}{2}, \\ q_l(Z) &= \frac{1}{\bar{r}} \inf_{\pi^* \gg 0} \mathbb{E}^*[Z] = \frac{1}{2} \inf_{\pi^* \in \Pi} \mathbb{E}^*[Z] = \frac{1}{2} \inf_{0 < \pi_3^* < 1/3} (3 - 5\pi_3^*) = \frac{2}{3}. \end{aligned}$$

3.4. Summary on "no arbitrage"

Theorem 3.25. *The following statements are equivalent:*

(1) *There is no arbitrage in the market model.*

(2) *The payoff pricing functional q is linear and strictly positive on $\mathcal{M}$.*

(3) *There exists a strictly positive valuation functional Q.*

(4) *There exists a vector $\bar{q} \in \mathbb{R}^S$ such that $P = X\bar{q}^T$ and $\bar{q} \gg 0$.*

(5) *There exists a risk-neutral probability (π_i^*) such that $\pi^* \gg 0$, where*

$$\pi_i^* = \frac{q_i}{\sum_{j=1}^{S} q_j}.$$

Similar results to no strong arbitrage.

3.5. Exercise

(1) Find the state price of the following valuation functionals.

(a) Two states and two securities with $P_1 = 1$, $X_1 = (1, 1)$, $P_2 = 1$, $X_2 = (2, 0.5)$.

(b) Three states and three securities with $P_1 = 1$, $X_1 = (2, 2, 2)$, $P_2 = 1$, $X_2 = (1, 2, 3)$, $P_3 = 1$, $X_3 = (4, 2, 1)$.

(2) Find a positive solution $\bar{q}$ to $P = X\bar{q}^T$ if it exists.

(a) Two states and one security with $P_1 = 1$, $X_1 = (2, 2)$.

(b) Three states and one security with $P_1 = 1$, $X_1 = (1, 2, 3)$.

(c) Three states and two securities with $P_1 = 1$, $X_1 = (2, 2, 2)$, $P_2 = 1$, $X_2 = (1, 2, 3)$.

(d) Three states and three securities with $P_1 = 1$, $X_1 = (2, 2, 2)$, $P_2 = 1$, $X_2 = (1, 2, 3)$, $P_3 = 1$, $X_3 = (4, 2, 1)$.

(3) Find the value of $q_u(Z)$ and $q_l(Z)$ for the following models.

(a) One security and three states with $P_1 = 1$, $X_1 = (1, 4, 2)$. A contingent claim with payoff $Z = (2, 1, 3)$.

(b) Two securities and three states with $P_1 = 1$, $X_1 = (2, 2, 2)$, $P_2 = 2$, $X_2 = (3, 4, 5)$. A contingent claim with payoff $Z = (4, 6, 2)$.

(c) Two securities and four states with $P_1 = 1$, $X_1 = (1, 1, 1, 1)$, $P_2 = 2$, $X_2 = (1, 2, 3, 4)$. A contingent claim with payoff $Z = (4, 3, 2, 1)$.

(d) Two securities and four states with $P_1 = 1$, $X_1 = (1, 1, 1, 1)$, $P_2 = 2$, $X_2 = (1, 2, 3, 4)$. A contingent claim with payoff $Z = (1, 1, 2, 2)$.

(4) Find the risk-neutral probability in the following cases.

(a) Two states and one security with $P_1 = 1$, $X_1 = (2, 2)$, $P_2 = 1$, $X_2 = (1, 4)$.

(b) Three states and two securities with $P_1 = 1$, $X_1 = (2, 2, 2)$, $P_2 = 1$, $X_2 = (1, 2, 3)$.

(c) Three states and three securities with $P_1 = 1$, $X_1 = (2, 2, 2)$, $P_2 = 1$, $X_2 = (1, 2, 3)$, $P_3 = 1$, $X_3 = (4, 2, 1)$.

(5) Consider a financial market with three states and three securities with $P_1 = 1$, $X_1 = (1, 1, 1)$, $P_2 = 1$, $X_2 = (0.8, 1, 2)$, $P_3 = 1$, $X_3 = (0.5, 2, 1)$.

(a) Find the risk-neutral probability π^*.

(b) Find $\mathbb{E}^*[Z_1]$ for contingent claim $Z_1 = (3, 4, 5)$.

(c) Find $\mathbb{E}^*[Z_2]$ for contingent claim $Z_2 = (1, 4, 10)$.

(6) Consider a financial market with three states and two securities with $P_1 = 1$, $X_1 = (1, 1, 1)$, $P_2 = 1$, $X_2 = (0.8, 1, 2)$.

(a) Find the collection of all possible risk-neutral probabilities.

(b) Find $q_u(1, 2, 3)$ and $q_l(1, 2, 3)$ using the above result.

(7) (a) Prove that there is "no arbitrage" in a financial model if and only if there exists a risk-neutral probability π^* and $\pi^* \gg 0$.

(b) Prove that there is "no strong arbitrage" in a financial model if and only if there exists a risk-neutral probability π^* and $\pi^* \geq 0$.