CHAPTER 6

Risk-Aversion

Consider two utility functions.

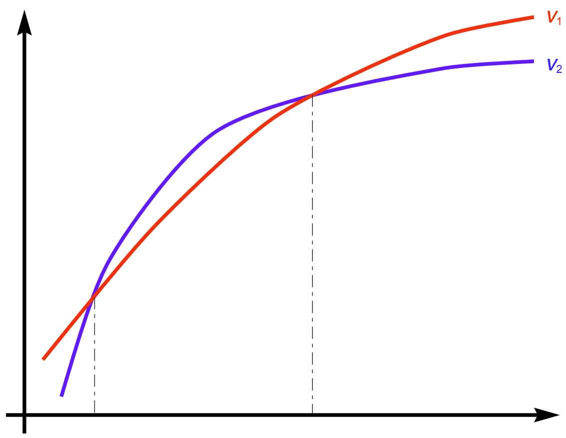

FIGURE 6.1

It seems that the utility function v_2 is risker than the utility function v_1. How is the logic? How can we say a utility function is riskier than the other one? Any measure of risk aversion must be local. We may look the two pictures in Figure 6.2. In the lower consumption level, the right is riskier than the left and in the higher consumption level, the left is riskier than the right.

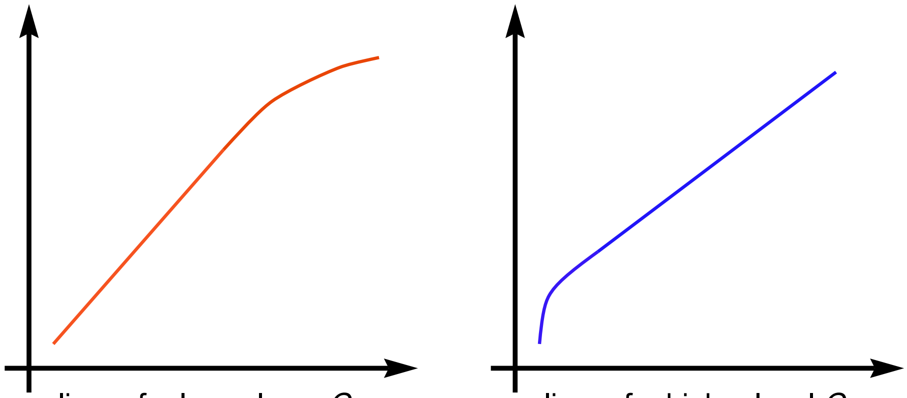

FIGURE 6.2

6.1. Arrow-Praft measures of risk aversion

Recall the von Neumann-Morgenstern utility function:

$$U(C_1, C_2, \cdots, C_S) \geq U(C_1', C_2', \cdots, C_S') \quad \Longleftrightarrow \quad \sum_{s=1}^{S} \pi_s u(C_s) \geq \sum_{s=1}^{S} \pi_s u(C_s')$$

Remark 6.1. Von Neumann-Morgenstern utility function is not unique. If we replace $u(\cdot)$ by $w(\cdot) = au(\cdot) + b$ with $a > 0$, then

$$\sum_s \pi_s u(C_s) \geq \sum_s \pi_s u(C_s') \quad \Longleftrightarrow \quad \sum_s \pi_s w(C_s) \geq \sum_s \pi_s x(C_s').$$

Question. How to measure the risk aversion?

Consider two utility functions $u(C)$ and $6 \times 10^{23} u(C)$,

Their second derivatives: $u''(C)$ and $6 \times 10^{23} u''(C)$.

We find that when we normalize $-u''(\cdot)$ by the first derivative, i.e.,

$$-\frac{u''(C)}{u'(C)},$$

then this representation is independent of affine transformation, i.e., after affine transformation

$$-\frac{u''(C)}{u'(C)} = -\frac{w''(C)}{w'(C)}.$$

In the following suppose $u'(C) \neq 0$,

Definition 6.2. (1) $-\dfrac{u''(C)}{u'(C)}$ is called the level of absolute risk aversion (ARA) of $u(\cdot)$ at C (or the Arrow-Pratt coefficient of absolute risk aversion of u at level C).

(2) $-\dfrac{Cu''(C)}{u'(C)}$ is called the level of relative risk aversion (RRA) of $u(\cdot)$ at C.

Notation 6.3. Denote

$$\text{ARA}(C) = -\frac{u''(C)}{u'(C)}, \qquad \text{and} \qquad \text{RRA}(C) = -\frac{Cu''(C)}{u'(C)}.$$

Example 6.4. (1) $u(x) = \ln x$, then

$$\begin{aligned} \text{ARA}(C) &= -\frac{u''(C)}{u'(C)} = \frac{-1/C^2}{1/C} = \frac{1}{C}, \\ \text{RRA}(C) &= C \cdot \text{ARA}(C) = 1. \end{aligned}$$

(2) $u(x) = \sqrt{x}$, then

$$\begin{aligned} \text{ARA}(C) &= -\frac{u''(C)}{u'(C)} = \frac{1}{2C}, \\ \text{RRA}(C) &= C \cdot \text{ARA}(C) = \frac{1}{2}. \end{aligned}$$

(3) $u(x) = \dfrac{x}{\ln x}$ for $x \geq e^2$. Then

$$u'(x) = \frac{\ln x - 1}{(\ln x)^2}, \qquad \text{and} \qquad u''(x) = \frac{2 - \ln x}{x(\ln x)^3}.$$

Thus,

$$\begin{aligned} \mathrm{ARA}(C) &= -\frac{u''(C)}{u'(C)} = \frac{\ln C - 2}{C \ln C(\ln C - 1)}, \\ \mathrm{RRA}(C) &= C \cdot \mathrm{ARA}(C) = \frac{\ln C - 2}{\ln C(\ln C - 1)}. \end{aligned}$$

Remark 6.5. If $\mathrm{ARA}(C) = -\dfrac{u''(C)}{u'(C)}$, then

$$\int_{C_1}^{C_2} \mathrm{ARA}(C)\, dC = -\int_{C_1}^{C_2} \frac{u''(C)}{u'(C)}\, dC = -\ln u'(C_2) + \ln u'(C_1)$$

Hence,

$$\ln\left(\frac{u'(C_2)}{u'(C_1)}\right) = -\int_{C_1}^{C_2} \mathrm{ARA}(C)\, dC.$$

This implies

$$u'(C_2) = u'(C_1) \exp\left(-\int_{C_1}^{C_2} \mathrm{ARA}(C)\, dC\right).$$

Example 6.6. (1) If $\mathrm{RRA}(C) = \gamma$, then $\gamma = -\dfrac{u''(C)}{u'(C)}$. This implies that

$$-\ln u'(C) + \ln u'(1) = -\int_1^C \frac{u''(x)}{u'(x)}\, dx = \int_1^C \frac{\gamma}{C}\, dx = \gamma \ln C,$$

i.e.,

$$u'(C) = u'(1) C^{-\gamma}.$$

Thus,

$$u(C) = \begin{cases} A \ln C + B, & \text{if } \gamma = 1, \\ A x^{1-\gamma} + B, & \text{if } \gamma \neq 1. \end{cases}$$

(2) If $\mathrm{ARA}(C) = \gamma$, then $\gamma = -\dfrac{u''(C)}{u'(C)}$. This implies that

$$-\ln u'(C) + \ln u'(1) = -\int_1^C \frac{u''(x)}{u'(x)}\, dx = \int_1^C \gamma\, dx = \gamma(C - 1),$$

i.e.,

$$u'(C) = u'(1) \exp\left(\gamma(1 - C)\right).$$

Thus,

$$u(C) = A\exp(-\gamma C) + B.$$

6.2. Risk compensation

Definition 6.7. We define the risk compensation as the amount of deterministic consumption one would have to charge an agent in exchange for relieving him or her of a risk. Explicitly, the risk compensation for the additional consumption plan Z at deterministic initial consumption y is the value $\pi(C,Z)$ that satisfies

$$E[u(C+Z)] = u(C - \pi(C,Z)) \tag{6.1}$$

(C is a constant and $E[Z] = 0$).

Remark 6.8. $\pi(C,Z)$ is also called the risk premium.

Remark 6.9.

(1) We may use another terminology to explain the risk premium. Since u is a strictly increasing continuous function, using the intermediate value theorem, there eixsts a unique number $c(\mu)$ such that

$$u(c(\mu)) = U(\mu) = \int u(x)\,\mu(dx). \tag{6.2}$$

It follows that

$$\delta_{c(\mu)} \sim \mu,$$

i.e., there is indifference between μ and the sure amount of money $c(\mu)$.

(2) The number $c(\mu)$ of (6.2) is called the certainty equivalent of the lottery $\mu \in \mathcal{M}$.

(3) Clearly, the number C is exactly the expectation of the the lottery μ, $m(\mu)$. Thus, the value

$$\pi(\mu) = m(\mu) - c(\mu)$$

is exactly the risk premium of μ.

(4) If u is strictly concave, the risk aversion implies via Jensen's inequality that $c(\mu) \leq m(\mu)$, and

$$c(\mu) < m(\mu) \qquad \Longleftrightarrow \qquad \mu \neq \delta_{m(\mu)}.$$

In particular, the risk premium $\pi(\mu)$ is strictly positive as soon as the distribution μ carries any risk.

(5) The certainty equivalent $c(\mu)$ can be viewed as an upper bound for any price of μ which would be acceptable to an economic agent with utility function u.

(6) The risk premium may be viewed as the amount that the agent would be ready to pay for replacing the asset by its expected value $m(\mu)$.

Example 6.10. (1) Suppose that $\mu = (1, 4, 9, 16, 25)$ with probability 1/5 at each state. Then the corresponding

$$\begin{aligned} C &= m(\mu) = 11 \\ Z &= (-10, -7, -2, 5, 14). \end{aligned}$$

For $u(x) = \sqrt{x}$, we have

$$\begin{aligned} U(\mu) &= \sqrt{1} \cdot \frac{1}{5} + \sqrt{4} \cdot \frac{1}{5} + \sqrt{9} \cdot \frac{1}{5} + \sqrt{16} \cdot \frac{1}{5} + \sqrt{25} \cdot \frac{1}{5} \\ &= 3 = u(c(\mu)) = \sqrt{c(\mu)}. \end{aligned}$$

Hence, $c(\mu) = 9$. Thus, the risk premium is given by

$$\pi(C, Z) = \pi(\mu) = m(\mu) - c(\mu) = 11 - 9 = 2.$$

(2) Suppose that $\mu = (1, 2, 4, 8, 16)$ with probability $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}. \frac{1}{16}, \frac{1}{16}\right)$. Then

$$\begin{aligned} C &= m(\mu) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + 8 \cdot \frac{1}{16} + 16 \cdot \frac{1}{16} = 3 \\ Z &= (-2, -1, 1, 5, 13). \end{aligned}$$

For $u(x) = \log x$, we have

$$\begin{aligned} U(\mu) &= \log 1 \cdot \frac{1}{2} + \log 2 \cdot \frac{1}{4} + \log 4 \cdot \frac{1}{8} + \log 8 \cdot \frac{1}{16} + \log 16 \cdot \frac{1}{16} \\ &= \log 2 \cdot \left(\frac{1}{4} + \frac{1}{4} + \frac{3}{16} + \frac{1}{4}\right) = \frac{15}{16} \log 2. \end{aligned}$$

Hence. $c(\mu) = 2^{15/16}$ and its risk premium

$$\pi(C, Z) = \pi(\mu) = m(\mu) - c(\mu) = 3 - 2^{15/16}.$$

Theorem 6.11. *For Z with $E[Z] = 0$, and h is small.*

$$\pi(C, hZ) \cong \frac{h^2}{2} ARA(C) \cdot \mathrm{Var}(Z).$$

Proof. Due to the Taylor expansion on the both sides of (6.1)

$$\begin{aligned} E[u(C + hZ)] &= E\left[u(C) + u'(C) \cdot hZ + \frac{1}{2} u''(C)(hZ)^2 + \cdots\right] \\ &= E[u(C)] + hE[u'(C)Z] + \frac{1}{2} h^2 E[u''(C)Z^2] + \cdots \\ u(C - \pi(C, hZ)) &= u(C) - \pi(C, hZ)u'(C) + \cdots, \end{aligned}$$

we have

$$u(C) - \pi(C, hZ)u'(C) + \cdots = \underbrace{E[u(C)]}_{=u(C)} + h \underbrace{E[u'(C)Z]}_{=0} + \frac{1}{2} h^2 E[u''(C)Z^2] + \cdots.$$

Thus,

$$-\pi(C, hZ)u'(C) \cong \frac{1}{2} h^2 u''(C)E[Z^2].$$

This implies

$$\pi(C, hZ) \cong \frac{1}{2} h^2 \left(-\frac{u''(C)}{u'(C)} \right) E[Z^2] = \frac{h^2}{2} \mathrm{ARA}(C) \mathrm{Var}(Z).$$

□

Remark 6.12. Using the terminology in Remark 6.9(3) we can rewrite the above theorem as

$$\pi(\mu) \cong \frac{1}{2} ARA(m(\mu)) \cdot \mathrm{Var}(\mu).$$

Definition 6.13. The relative risk compensation for the relative risk Z at deterministic initial consumption C is the value $\pi_R(C, Z)$ that satisfies

$$E[u(C + CZ)] = u(C - C\pi_R(C, Z)).$$

Proposition 6.14. *For Z with $E[Z] = 0$, and h is small, we have*

$$\pi_R(C, hZ) \cong \frac{h^2}{2} RRA(C) \cdot \mathrm{Var}(Z).$$

PROOF. Exercise. □

6.3. The Pratt theorem

The two measures of risk aversion - the Arrow-Pratt measure and risk compensation - can be used to compare the risk aversion of two agents. An important theorem says that the comparison using the Arrow-Pratt and risk compensation always give the same result.

Let u_1 and u_2 be two von Neumann-Morgenstern utility functions on $\mathbb{R}$, and let π_i and ARA_i denote the risk compensation and the Arrow-Pratt measure of absolute risk aversion, respectively, of u_i for $i = 1, 2$.

Theorem 6.15 (Pratt Theorem). *Suppose the utility functions $u_1, u_2 \in C^2$ are strictly increasing. Then the following conditions are equivalent:*

(1) $ARA_1(C) \geq ARA_2(C)$ *for every* C.

(2) $\pi_1(C, Z) \geq \pi_2(C, Z)$ *for every constant* C *and every random variable* Z.

(3) u_1 *is a concave transformation of* u_2*; that is,* $u_1 = f \circ u_2$ *for* f *concave and strictly increasing.*

PROOF. "(1) $\Longrightarrow$ (3)": Since u_2 is strictly increasing, its inverse function u_2^{-1} exists, and we can define the function f by $f(t) = u_1(u_2^{-1}(t))$. Remain to show that f is strictly increasing and concave. The first derivative of f is give by

$$f'(t) = \frac{u_1'(u_2^{-1}(t))}{u_2'(u_2^{-1}(t))},$$

which is strictly positive, since $u_i' > 0$ for $i = 1, 2$. The second derivative of f yields

$$f''(t) = \frac{u_1''(y)u_2'(y) - u_2''(y)u_1'(y)}{(u_2'(y))^3} = (ARA_2(y) - ARA_1(y))\,\frac{u_1'(y)}{(u_2'(y))^2},$$

where $y = u_2^{-1}(t)$. Thus $f'' < 0$, and hence f is concave.

"(3) $\Longrightarrow$ (2)": By the definition of the risk compensation we have

$$E[u_1(C + Z)] = u_1(C - \pi_1(C, Z)).$$

Because $u_1 = f \circ u_2$ and f is concave, application of Jensen's inequality yields

$$\begin{aligned} E[u_1(C + Z)] &= E[f(u_2(C + Z))] \leq f\left(E[u_2(C + Z)]\right) \\ &= f(u_2(C - \pi_2(C, Z))). \end{aligned}$$

Combining the above two inequalities, we have

$$u_1(C - \pi_1(C, Z)) \leq u_1(C - \pi_2(C, Z)).$$

Since u_1 is strictly increasing, thus $\pi_1(C,Z) \geq \pi_2(C,Z)$.

"(2) $\Longrightarrow$ (1)": Suppose that $ARA_1(C^*) < ARA_2(C^*)$ for some C^*. Since ARA_1 and ARA_2 are continuous, there is an interval around C^* such that $ARA_1(C) < ARA_2(C)$ for every C in this interval. By Theorem 6.11, if h is small enough, then

$$\pi_1(C,hZ) < \pi_2(C,hZ),$$

which contradicts to the second assertion. □

Remark 6.16. The following conditions are equivalent:

(1) $ARA_1(C) > ARA_2(C)$ for every C.

(2) $\pi_1(C,Z) > \pi_2(C,Z)$ for every C and every random variable Z.

(3) u_1 is a concave transformation of u_2; that is, $u_1 = f \circ u_2$ for f strictly concave and strictly increasing.

Corollary 6.17. *If u is a strictly concave C^2 utility function, then*

(1) *$\pi(C,Z)$ is increasing in C for every Z $\iff$ $ARA(C)$ is increasing in C.*

(2) *$\pi(C,Z)$ is constant in C for every Z $\iff$ $ARA(C)$ is constant in C.*

(3) *$\pi(C,Z)$ is decreasing in C for every Z $\iff$ $ARA(C)$ is decreasing in C.*

Proof. (1) Define a utility function v by

$$v(C) = u(C + \Delta C)$$

for some $\Delta C > 0$, then the Arrow-Pratt measure of absolute risk aversion is of the form

$$ARA_v(C) = ARA(C + \Delta C),$$

and the risk premium of v is given by

$$\pi_v(C,Z) = \pi(C + \Delta C, Z).$$

Applying Theorem 6.15, we have

$$ARA(C + \Delta C) = ARA_v(C) > ARA(C) \quad \Longleftrightarrow \quad \pi(C + \Delta C, Z) = \pi_v(C, Z) = \pi(C, Z).$$

Since ΔC is arbitrary, (1) follows.

(2), (3): Exercise. □

6.4. Examples of utility functions

(1) Quadratic utility:

$$u(C) = AC - BC^2, \qquad \text{if } A - 2BC \geq 0.$$

It looks very nice, but we should almost immediately forget about it again.

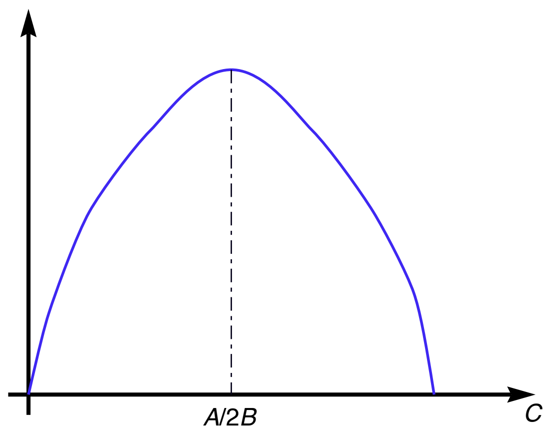

FIGURE 6.3. quadratic utility function

Problem: Image is bounded.

$$\text{ARA}(C) = \frac{2B}{A - 2BC}: \qquad \text{increasing in } C.$$

Advantage (?): expected utility depends only on $E[C]$ and $\mathrm{Var}(C)$, since

$$\begin{aligned} E[u(C)] &= AE[C] - BE[C^2] \\ &= AE[C] - B((E[C])^2 + \mathrm{Var}(C)). \end{aligned}$$

(2) Negative exponential utility: (Constant absolute risk aversion (CARA) or hyperbolic absolute risk aversion (HARA))

$$u(C) = -\exp(-\rho C).$$

It does not matter when the utility is negative. Sometimes it is non-realistic. However, without CARA it is difficult to consider and discuss some economic properties.

$$\mathrm{ARA}(C) = -\frac{-\rho^2 e^{-\rho C}}{\rho\, e^{-\rho C}} = \rho = \text{ constant}.$$

These utility functions can be shifted to any interval (a, ∞).

(3) Power utility and logarithmic utility function: (Constant relative risk aversion (CRRA))

$$u(C) = \begin{cases} \dfrac{C^\gamma}{\gamma}, & \text{if } \gamma \neq 0, \gamma < 1, \\ \log C, & \text{if } \gamma = 1. \end{cases}$$

Then $u'(C) = C^{\gamma-1}$.

$$\begin{aligned} \mathrm{ARA}(C) &= -\frac{(\gamma-1)C^{\gamma-2}}{C^{\gamma-1}} = (1-\gamma)C^{-1} = \frac{1-\gamma}{C}, \\ \mathrm{RRA}(C) &= C \cdot \mathrm{ARA}(C) = 1 - \gamma. \end{aligned}$$

Remark 6.18. Logarithmic and negative exponential utility can be viewed as limiting cases of power utility when γ approaches 1 or 0.

6.5. Other measure of risk aversion

Definition 6.19. J is said to be strongly more risk averse than M (J, M: individuals) if

$$\inf_z \frac{u_J''(z)}{u_M''(z)} \geq \sup_z \frac{u_J'(z)}{u_M'(z)}.$$

Lemma 6.20. *J is strongly more risk averse than M, then*

$$ARA_J(z) \geq ARA_M(z) \qquad \textit{for all } z.$$

Proof. If J is strongly more risk averse than M, then

$$\inf_z \frac{u_J''(z)}{u_M''(z)} \geq \sup_z \frac{u_J'(z)}{u_M'(z)}.$$

Thus,

$$\frac{u_J''(z)}{u_M''(z)} \geq \frac{u_J'(z)}{u_M'(z)},$$

i.e.,

$$-\frac{u_J''(z)}{u_J'(z)} \leq -\frac{u_M''(z)}{u_M'(z)}.$$

This inequality is exactly the inequality $ARA_J(z) \geq ARA_M(z)$. □

But the converse is not true.

Example 6.21. Let

$$u_J(z) = -e^{-az} \qquad \text{and} \qquad u_M(z) = -e^{-bZ}$$

with $a > b$. Thus, $ARA_J(z) \geq ARA_M(z)$. However,

$$\begin{aligned}
\frac{u_J'(z)}{u_M'(z)} &= \frac{ae^{-az}}{be^{-bz}} = \frac{a}{b}\, e^{(b-a)z}, \\
\frac{u_J''(z)}{u_M''(z)} &= \frac{-a^2e^{-az}}{-b^2e^{-bz}} = \frac{a^2}{b^2}\, e^{(b-a)z}.
\end{aligned}$$

If $z_2 - z_1$ is large,

$$\frac{u_J''(z)}{u_M''(z)} < \frac{u_J'(z)}{u_M'(z)},$$

i.e., J is not strongly more risk averse than M.

Proposition 6.22. *J is strongly more risk averse than M $\iff$ there exists a decreasing concave function G and $\lambda > 0$ such that*

$$u_J(z) = \lambda u_M(z) + G(z) \qquad \textit{for all } z.$$

Proof. "$\Longleftarrow$": If $u_J(z) = \lambda u_M(z) + G(z)$, then

$$\begin{aligned} u_J'(z) &= \lambda u_M'(z) + G'(z) \\ u_J''(z) &= \lambda u_M''(z) + G''(z) \end{aligned}$$

for all z.

$$\begin{aligned} \frac{u_J''(z_2)}{u_M''(z_2)} &= \lambda + \frac{G''(z_2)}{u_M''(z_2)} \underset{G\text{:concave}}{\geq} \lambda \underset{G\text{:decreasing}}{\geq} \lambda + \frac{G'(z_1)}{u_M'(z_1)} \\ &= \frac{u_J'(z_1)}{u_M'(z_1)} \end{aligned}$$

for all z_1, z_2. This implies

$$\inf_z \frac{u_J''(z)}{u_M''(z)} \geq \sup_z \frac{u_J'(z)}{u_M'(z)}.$$

"$\Longrightarrow$" Suppose that J is strongly more risk averse than M, then for all z, there exists a $\lambda > 0$ such that

$$\frac{u_J''(z)}{u_M''(z)} \geq \lambda \geq \frac{u_J'(z)}{u_M'(z)}.$$

Define

$$G(z) = u_J(z) - \lambda u_M(z).$$

Since

$$G'(z) = u_J'(z) - \lambda u_M'(z) \leq 0,$$

G is decreasing. Moreover, by

$$G''(z) = u_J''(z) - \lambda u_M''(z) \leq 0,$$

G is concave. □

6.6. Exercise

(1) Calculate the absolute risk aversion (ARA) and the relative risk aversion (RRA) of the following utility functions:

(a) (narrow power utility function) $u(C) = \dfrac{B}{B-1} C^{1-\frac{1}{B}}$ for $C > 0$, $B > 0$;

(b) (extended power utility function) $u(C) = \dfrac{1}{B-1}(A + BC)^{1-\frac{1}{B}}$ for $B > 0$, $A \neq 0$, $C > \max\left(-\dfrac{A}{B}, 0\right)$.

(2) Find the conditions on which the relative risk aversion (RRA) of the extended power utility function is decreasing.

(3) Let δ be a non-zero constant.

(a) Suppose the absolute risk aversion (ARA) of a utility function is given by δ, find the utility function;

(b) Suppose the relative risk aversion (RRA) of a utility function is given by δ, find the utility function;

(c) Suppose the ARA of a utility function is given by $\mathrm{ARA}(C) = \exp(-C)$ and $u(0) = u'(0) = 1$, find the utility function.

(4) Find the risk compensation (risk premium) in the following models.

(a) the von Neumann-Morgenstern utility function $u(x) = -x^2$, initial wealth $C = 1$, profit $Z = (1, -1)$, probability $p = (1/2, 1/2)$;

(b) the von Neumann-Morgenstern utility function $u(x) = -\exp(-x)$, profit $(3\ln 2, 0)$, probability $p = (1/3, 2/3)$;

(c) the von Neumann-Morgenstern utility function $u(x) = -\exp(-x)$, initial wealth $C = 0$, profit Z is a standard normal distribution (i.e., $Z \sim N(0,1)$).

(5) Compute the absolute risk aversion and relative risk aversion of the following utility functions. Moreover, find the relation of the utility functions in the sense of "strongly more risk aversion".

$$u_1(x) = x;$$
$$u_2(x) = x^{1/3};$$
$$u_3(x) = x^{1/3} - x;$$
$$u_4(x) = 100x^{1/3} - x.$$