Chapter 2
Fourier series and boundary value problems

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- In this chapter, we introduce the useful tool – Fourier series and show how to use them to solve boundary value problems for PDEs.

(mainly 2nd order)

2-1 Fourier series

- In these two sections, we will study the following trigonometric series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

$\Rightarrow$ These series are called **Fourier series**.

$$T = \frac{2L}{m} \ : \ \text{period}$$

- **Goal**：

1) To determine what functions can be represented as a sum of a Fourier series.

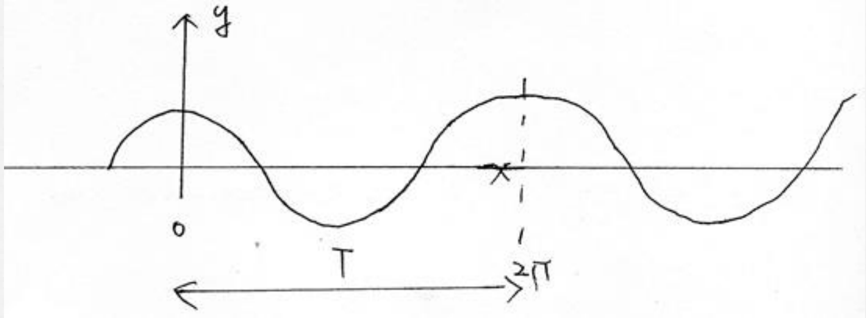
2) To compute the coefficients.

3) The convergence of the series.

- **Def**：

f is said to be periodic with period $T > 0$ if the domain of f contains $x + T$ whenever it contains x, and if $f(x+T) = f(x)$ for every value of x.

The smallest value of T is called the **fundamental period** of f.



Orthogonality of the sine and cosine functions.

- Let u, v be two real – valued functions on the interval $\alpha \le x \le \beta$, we can define the standard inner product $\langle u, v\rangle = \int_{\alpha}^{\beta} u(x)v(x)\,dx$.
- The functions u and v are said to be orthogonal on $\alpha \le x \le \beta$ if $\int_{\alpha}^{\beta} u(x)v(x)\,dx = 0$.
- A set of functions is said to be mutually orthogonal if each distinct pair of functions in the set is orthogonal.

- **Thm**：

The functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, \cdots$, form a mutually orthogonal set of functions on $-L \le x \le L$ and

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx = \begin{cases} 0, & m \ne n \\ L, & m = n \end{cases}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = 0, \qquad \forall\, m, n$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = \begin{cases} 0, & m \ne n \\ L, & m = n \end{cases}$$

proof：

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\ dx$$

$$=\frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(m-n)\pi x}{L}-\cos\frac{(m+n)\pi x}{L}\right]dx$$

$$=\frac{1}{2}\frac{L}{\pi}\left\{\sin\frac{(m-n)\pi x/L}{m-n}-\sin\frac{(m+n)\pi x/L}{m+n}\right\}\Bigg|_{-L}^{L}$$

$= 0$ if $m \neq n,\ \ m+n \neq 0.$

$m = n$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\; dx$$

$$= \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)^2 dx$$

$$= \frac{1}{2}\int_{-L}^{L} \left[1 - \cos\frac{2mx}{L}\right] dx$$

$$= L$$

- **The Euler – Fourier Formula**

Suppose

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

converges uniformly to $f(x)$ on $[-L, L]$, then

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 0, 1, 2, \cdots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, 3, \cdots$$

proof：

From our assumption,

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

$$\text{——} (*)$$

(1) To find a_0, we integrate ($\ast$) from – L to L and obtain

$$\int_{-L}^{L} f(x)\,dx$$

$$= \frac{a_0}{2}\int_{-L}^{L} dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos\frac{m\pi x}{L}\,dx$$

$$+ \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin\frac{m\pi x}{L}\,dx$$

$$= La_0$$

$$\Rightarrow\ a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

(2) To find a_n, $\mathrm{n} = 1, 2, 3, \cdots$, we multiply $(\ast)$ by $\cos\frac{n\pi x}{L}$ and integrate from $-L$ to L

$$\Rightarrow \int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{a_0}{2}\int_{-L}^{L}\cos\frac{n\pi x}{L}dx$$

$$+\sum_{m=1}^{\infty} a_m \int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx$$

$$+\sum_{m=1}^{\infty} b_m \int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx$$

By the orthogonality relations,

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = La_n, \qquad \mathrm{n} = 1, 2, \cdots$$

$$\therefore\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad \mathrm{n} = 1, 2, \cdots$$

(3) b_n can be found similarly by multiplying

$(\ast)$ by $\sin\frac{n\pi x}{L}$ and integrating from – L to L

$$\Rightarrow\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \qquad \mathrm{n} = 1, 2, 3, \cdots$$

Ex：

$$f(x)=\begin{cases}-x, & -2\le x\le 0\\ x, & 0\le x\le 2\end{cases},$$

$$f(x+4)=f(x)$$

Determine the coefficients in the Fourier series of $f(x)$.

<u>sol</u>：

$L=2.$

$\therefore$ The Fourier series has the form

$$f(x)=\frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m\cos\frac{m\pi x}{2}+b_m\sin\frac{m\pi x}{2}\right)$$

To find a_n, we compute

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} (-x)\,dx + \frac{1}{2}\int_{0}^{2} x\,dx = 2$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} -x\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[-\frac{2}{n\pi} x sin\ \frac{n\pi x}{2} - \left(\frac{2}{n\pi}\right)^2 \cos\frac{n\pi x}{2}\right]_{-2}^{0}$$

$$+\frac{1}{2}\left[-\left(\frac{2}{n\pi}\right)^2 + \left(\frac{2}{n\pi}\right)^2 \cos n\pi + \left(\frac{2}{n\pi}\right)^2 \cos n\pi - \left(\frac{2}{n\pi}\right)^2\right]$$

$$= \frac{4}{(n\pi)^2}(\cos n\pi - 1), \qquad n = 1, 2, \cdots$$

$$= \begin{cases} \dfrac{-8}{(n\pi)^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

To compute b_n,

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx, \qquad \text{n} = 1, 2,3, \cdots$$

$$\because\ f \text{ is even,} \qquad \sin\frac{n\pi x}{2} \text{ is odd.}$$

$$\therefore\ f(x)\sin\frac{n\pi x}{2} \text{ is odd.}$$

$$\Rightarrow\ \int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = 0$$

$$\Rightarrow\ b_n = 0\ \ \forall n$$

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{2} + b_m \sin\frac{m\pi x}{2}\right)$$

$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

$$= 1 - \frac{8}{\pi^2}\sum_{m=1,3,5,\cdots}^{\infty}\frac{\cos(m\pi x/2)}{m^2}$$

$$= 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x/2}{(2n-1)^2}$$

Ex：

$$f(x) = \begin{cases} 0, & -3 < x < -1 \\ 1, & -1 < x < 1 \\ 0, & 1 < x < 3 \end{cases},$$

$$f(x+6) = f(x)$$

Find the Fourier series of $f(x)$.

<u>sol</u>：

f has period 6, $\Rightarrow\ L = 3$.

$\therefore$ The Fourier series for f has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{3} + b_n \sin\frac{n\pi x}{3} \right)$$

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{2}{3}$$

$$a_n = \frac{1}{3}\int_{-1}^{1} \cos\frac{n\pi x}{3}\,dx = \frac{1}{n\pi} sin\ \frac{n\pi x}{3}\Big|_{-1}^{1}$$

$$= \frac{2}{n\pi} sin\ \frac{n\pi}{3}\ , \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{3}\int_{-1}^{1} \sin\frac{n\pi x}{3}\,dx = -\frac{1}{n\pi}\cos\frac{n\pi x}{3}\Big|_{-1}^{1}$$

$$= 0, \qquad \mathrm{n} = 1, 2{,}3, \cdots$$

Thus the Fourier series for f is

$$f(x)$$

$$= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n\pi} sin\ \frac{n\pi}{3} \cos \frac{n\pi x}{3}$$

$$= \frac{1}{3} + \frac{\sqrt{3}}{\pi} \left[\cos \frac{\pi x}{3} + \frac{\cos(2\pi x/3)}{2} \right.$$

$$\left. - \frac{\cos(4\pi x/3)}{4} - \frac{\cos(5\pi x/3)}{5} + \cdots \right]$$

2-2 The Fourier Convergence Theorem

- Let $f(x)$ be periodic with period $2L$ and integrable on $[-\mathrm{L}, \mathrm{L}]$, then all the Fourier coefficients can be computed, and the Fourier series can be written as

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L} \right)$$

$$\begin{aligned} a_m &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{m\pi x}{L} dx, \quad n = 0, 1, 2, \cdots \\ b_m &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\frac{m\pi x}{L} dx, \quad n = 1, 2, 3, \cdots \end{aligned} \qquad (*)$$

Q：

(1) Is the Fourier series convergent ?

(2) Is the sum equal to $f(x)$?

- To guarantee convergence of a Fourier series to the function, we need to impose extra condition on $f(x)$.

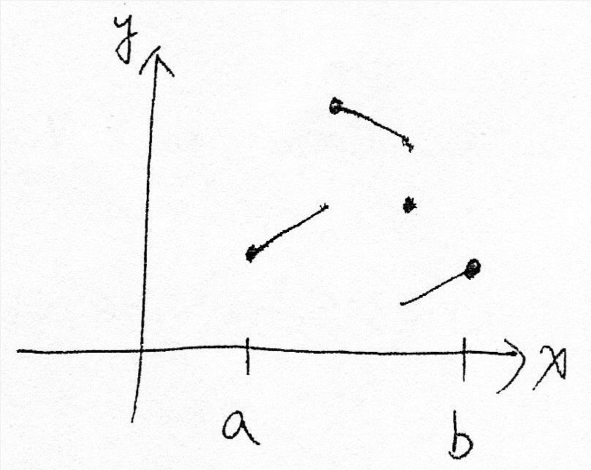
- **Def**：

A function $f(x)$ is said to be piecewise continuous on $a \leq x \leq b$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < \cdots < x_n = b$

so that

(1) f is continuous on each open subinterval $x_{i-1} < x < x_i$

(2) f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.



- Notation：

$$f(c+) = \lim_{x \to c+} f(x),$$

$$f(c-) = \lim_{x \to c-} f(x)$$

- Note that the function may not be defined at the partition points.

Thm：

Suppose that f and f' are piecewise continuous on the interval $-L \le x \le L$.

Furthermore, suppose that f is defined outside the interval $-L \le x \le L$ so that it is periodic with period $2L$.

Then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

whose coefficients are given by ($*$).

The Fourier series converges to $f(x)$ at all points where f is continuous, and to $\dfrac{f(x^+)+f(x^-)}{2}$ at all points where f is discontinuous.

Note that $\dfrac{f(x^+)+f(x^-)}{2}$ is the average value of the right−hand and left−hand limits at the point x.

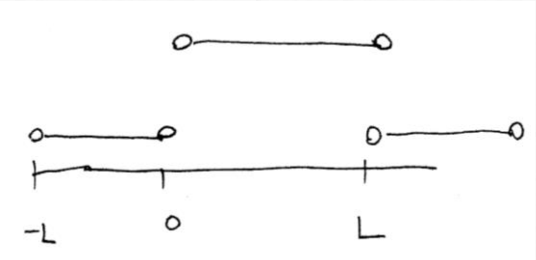
- Functions that are not included in the theorem are primarily those with infinite discontinuities in $[-L, L]$, such as $\frac{1}{x^2} \to 0 \; (x \to 0)$ or $ln|x - L| \; (x \to L)$.
- Functions having an infinite number of jump discontinuities in the interval are also excluded.

- **Remark**：

A Fourier series may converge to a sum that is not differentiable, or even continuous.

Ex：

$$f(x) = \begin{cases} 0, & -L < x < 0 \\ L, & 0 < x < L \end{cases}$$



We can always extend f by $f(x+2L) = f(x)$.

It is clear that $f(x)$ satisfies the hypothesis of Theorem.

⇒ We can compute the Fourier series of $f(x)$.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx = \int_0^L dx = L$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\, dx = \frac{1}{L}\int_0^L L\cos\frac{m\pi x}{2}\, dx$$

$$= 0, \qquad m \neq 0$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\, dx = \int_0^L \sin\frac{m\pi x}{L}\, dx$$

$$= \frac{L}{m\pi}(1 - \cos m\pi) = \begin{cases} 0, & m \text{ even} \\ \dfrac{2L}{m\pi}, & m \text{ odd} \end{cases}$$

$$\therefore\ f(x)$$

$$= \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{3} + b_m \sin\frac{m\pi x}{3}\right)$$

$$= \frac{L}{2} + \frac{2L}{\pi}\left(\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \cdots\right)$$

$$= \frac{L}{2} + \frac{2L}{\pi}\sum_{m\ \mathrm{odd}} \frac{\sin(m\pi x/L)}{m}$$

$$= \frac{L}{2} + \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x/L}{2n-1}$$

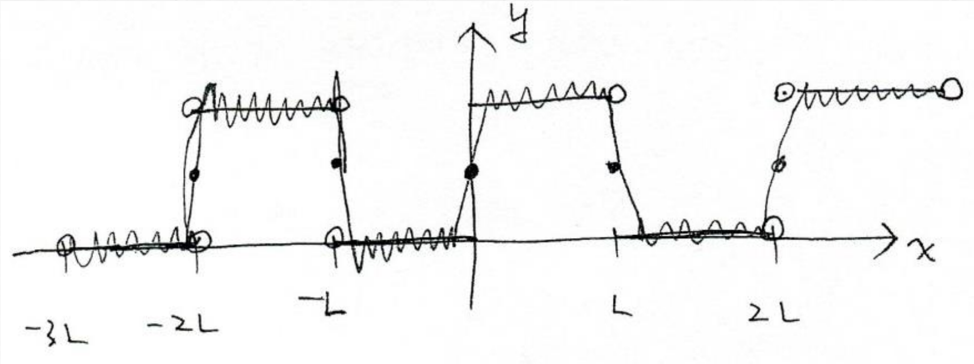
Note that at the discontinuities $(x = 0, \pm nL)$,

the Fourier series equals $\frac{L}{2}$.

This is exactly the mean value of right and left limit.

If we define $f(x) = \frac{L}{2}$ at $x = 0, \pm nL$,

then the Fourier series converges at all points x.



- Now we use a few examples to illustrate the idea of the fourier series.
- First we study the following ODE.

Ex1：

Find the solution of the initial value problem

$$\begin{cases} y'' + \omega^2 y = \sin nt \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

sol：

Case 1： $\omega^2 \neq n^2$

First we solve the homogeneous problem

$$y'' + \omega^2 y = 0$$

$\Rightarrow \mathrm{y(t)} = c_1 \sin \omega t + c_2 \cos \omega t$

Now to find a particular solution for the nonhomogeneous equation, we let

$$Y(t) = A\sin nt$$

$y'' + \omega^2 y = -An^2 \sin nt + \omega^2 A\sin nt = \sin nt$

$\therefore\ (\omega^2 - n^2) A\sin nt = \sin nt$

$$\therefore \ A = \frac{1}{\omega^2 - n^2}$$

$\therefore$ The general solution of the non - homogeneous problem can be written as

$$y(t) = c_1 \sin \omega t + c_2 \cos \omega t + \frac{1}{\omega^2 - n^2} \sin nt$$

From the initial conditions

$$y(0) = c_2 = 0$$

$$y'(t) = c_1\omega\cos\omega t - c_2\omega\sin\omega t + \frac{n}{\omega^2 - n^2}\cos nt$$

$$y'(0) = c_1\omega + \frac{n}{\omega^2 - n^2} = 0$$

$$\therefore\ c_1 = \frac{-n}{\omega(\omega^2 - n^2)}$$

$$\therefore\ y(t) = \left[\frac{-n\sin\omega t}{\omega(\omega^2 - n^2)} + \frac{\sin nt}{\omega^2 - n^2}\right] = \frac{\omega\sin nt - n\sin\omega t}{\omega(\omega^2 - n^2)}$$

Case 2：

if $\omega^2 = n^2$, then we solve

$$y'' + n^2 y = \sin nt$$

The particular solution we are looking for is

$$Y(t) = At\sin nt + Bt\cos nt$$

$$Y'(t) = A\sin nt + Ant\cos nt + B\cos nt - Bnt\sin nt$$

$$Y''(t) = An\cos nt + An\cos nt - An^2 t\sin nt - Bn\sin nt - Bn\sin nt - Bn^2 t\cos nt$$

$$Y''(t) + n^2 y$$
$$= 2An\cos nt - An^2 t\sin nt - 2Bn\sin nt$$
$$- Bn^2 t\cos nt + An^2 t\sin nt + Bn^2 t\cos nt$$
$$= 2An\cos nt - 2Bn\sin nt = \sin nt$$
$$An = 0, \qquad Bn = -\frac{1}{2}, \qquad B = -\frac{1}{2n}$$
$$\therefore\ Y(t) = -\frac{1}{2n} t\cos nt$$

$\therefore$. The general solution of the nonhomogeneous problem is

$$y(t) = c_1 \sin nt + c_2 \cos nt - \frac{1}{2n} t\cos nt$$

$$y(0) = c_2 = 0$$

$$y'(t) = c_1 n \cos nt - c_2 n \sin nt - \frac{1}{2n} \cos \mathrm{n}t$$
$$+ \frac{n}{2n} t \sin nt$$

$$y'(0) = c_1 n - \frac{1}{2n} = 0 \quad \Rightarrow \; c_1 = \frac{1}{2n^2}$$

$$\therefore \; y(t) = \frac{1}{2n^2} \sin nt - \frac{1}{2n} t\cos nt$$

Ex：

$$\begin{cases} y'' + \omega^2 y = f(t) \\ \quad y(0) = 0 \\ \quad y'(0) = 0 \end{cases} \quad , f(t) = \begin{cases} -1, & -\pi < t < 0 \\ 0, & t = -\pi, 0, \pi \\ 1, & 0 < t < \pi \end{cases}$$

<u>sol</u>：

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt, \qquad n = 0, 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt, \qquad n = 1, 2, \cdots$$

$\because f$ is odd, $\quad \therefore a_n = 0$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} \sin nt\,dt \\
&= \frac{1}{\pi}\frac{\cos nt}{n}\bigg|_{-\pi}^{0} - \frac{1}{\pi}\frac{\cos nt}{n}\bigg|_{0}^{\pi} \\
&= \frac{1}{n\pi} - \frac{\cos n\pi}{n\pi} - \frac{\cos n\pi}{n\pi} + \frac{1}{n\pi} \\
&= \frac{2}{n\pi} - \frac{2\cos n\pi}{n\pi} = \begin{cases} \dfrac{4}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}
\end{aligned}
$$

$$\therefore\ f(t)=\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin(2n-1)t$$

$\therefore$. The problem becomes

$$\begin{cases} y''+\omega^2 y=f(t)=\displaystyle\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin(2n-1)t \\ y(0)=0 \\ y'(0)=0 \end{cases} \qquad (*)$$

From the **Ex1**, the special solution for $y''+\omega^2 y=\sin nt$ is

$$Y(t)=\frac{1}{\omega^2-n^2}\sin nt \text{ for } \omega^2\neq n^2.$$

Therefore, the special solution for the nonhomogeneous equation in (∗) can be written by

$$Y(t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\pi} \frac{1}{\omega^2 - (2n-1)^2} \sin(2n-1)t$$

⇒ The general solution is

$$y(t) = c_1 \sin \omega t + c_2 \cos \omega t + Y(t).$$

From the initial condition, $y(0) = c_2 = 0$.

$\because \; y'(t) = \omega c_1 \cos \omega t + Y'(t)$

$\therefore \; y'(0) = c_1 \omega + Y'(0) = 0$

$$\because \; Y'(t) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{\omega^2 - (2n-1)^2} \cos(2n-1)t$$

$$\Rightarrow \; Y'(0) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{\omega^2 - (2n-1)^2}$$

$$\therefore \; c_1 = -\frac{1}{\omega} \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{\omega^2 - (2n-1)^2}$$

⇒ The solution of the IVP ($*$) is given by

$$y(t) = c_1 \sin \omega t$$

$$+ \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \frac{1}{\omega^2 - (2n-1)^2} \sin(2n-1)t$$

where $c_1 = -\frac{1}{\omega}\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{\omega^2-(2n-1)^2}$

Q： What about the case $\omega^2 = n^2$?

(Exercise)