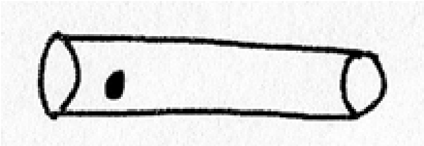
# 1-4 Initial and Boundary conditions

In ODE, the problem can have many solutions if we don't impose additional conditions.

The usual conditions we impose are initial and boundary conditions as we saw in ODE.

Ex： Diffusion equation

$$\begin{cases} u_t = k u_{xx} \\ u(x,t_0) = \phi(x) \end{cases}$$



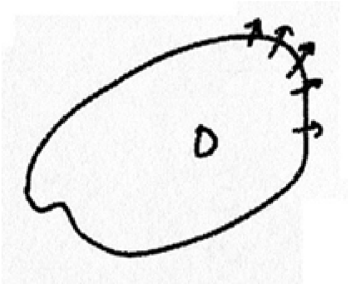
An initial condition specifies the physical state at a particular time $t_0$ .

For the diffusion equation, $\phi(x)$ is the initial concentration of the dye.

- Sometimes we restrict ourselves to a bounded domain D, in such case we need to impose boundary conditions.

We still consider the case of diffusion.

If the dye is enclosed in a container D so that none can escape or enter.




⇒ Nothing is going out or in from the boundary.

∴ u(concentration of the dye) is constant along the normal direction.

∴ $\frac{\partial u}{\partial n} = 0 = \nabla u \cdot n$ on $\partial D$

⇒ This is the common boundary condition — Neumann condition.

∗ Three most important kinds of boundary conditions：

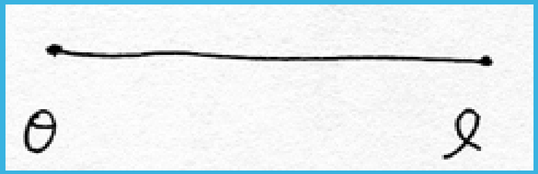
(D) u is specified. (Dirichlet condition)

(N) the normal derivative is specified. (Neumann condition)

(R) $\frac{\partial \mathrm{u}}{\partial \mathrm{n}} = au$ is specified. (Robin condition)

Ex： For the vibrating string$u_{tt} = cu_{xx}$

- If a string is held fixed at both ends, as for a violin string, we have the homogeneous Dirichlet conditions $(\theta, t) = u(\ell, t) = 0$ .



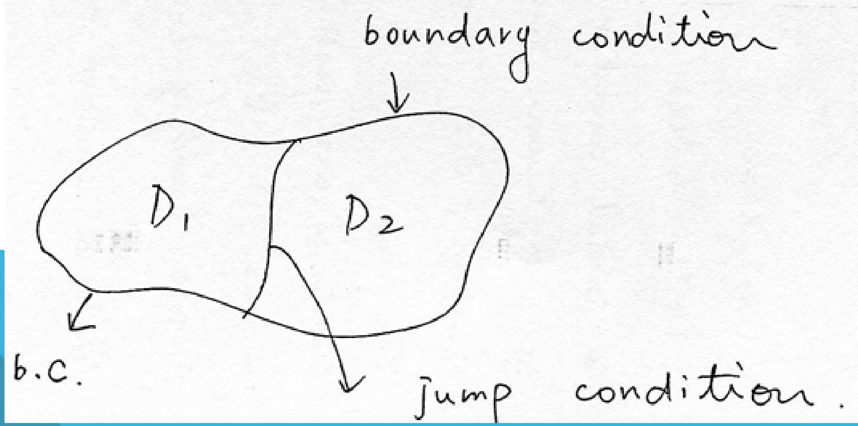
- If one end of the string is free to move transversally without any resistance, then there is no tension T at that end, so $u_x = 0$ . This is a Neumann condition.
- conditions at infinity

  If the domain D is unbounded, we usually provides conditions at infinity.

  For example, $u \to 0$ as $\to \infty$ .

- jump conditions

  If the domain D has two parts, $D = D_1 \cup D_2$ with different physical properties, for example, for heat conduction, where $D_1$ & $D_2$ consist of two different materials.



# 1-5 Well – posed problems

Many phenonmanon can be formulated by PDEs, like diffusion equation, wave equation & heat equations.

How do we know if the model really makes sense？

And which conditions we have to impose？

So we have the concept of wall – posed problems.

Q： What are well – posed problems？

Well – posed problems consist of a PDE in a domain together with a set of initial & boundary conditions (or other conditions) that satisfies the following fundamental properties：

1) Existence： There exists at least one solution $u(x,t)$ satisfying all the conditions.

2) Uniqueness： There is at most one solution.

3) Stability： The unique solution depends in a stable manner on the data of the problem.

i.e.： if the data are changed a little, the corresponding solution changes only a little.

- underdetermined – nonuniqueness
- overdetermined – nonexistence

The stability property is required in models of physical problems.
Because there is also an error in an experiment.
If tiny changes in the conditions result in big changes in the solution, then it is not good for experiment.

In PDE, the well – posedness problems are much harder to distinguish than ODEs.

Let us look at an example：

Ex：

Consider the Cauchy problem for the Laplace equation

$$\begin{cases} u_{xx} + u_{yy} = 0 \quad \rightarrow \quad elliptic \quad equation \\ u(o, y) = 0 \\ u_x(0, y) = \underbrace{\dfrac{1}{n}}_{small} \sin ny \end{cases} \xrightarrow{n \rightarrow \infty} \begin{cases} u_{xx} + u_{yy} = 0 \\ u(o, y) = 0 \\ u_x(0, y) = 0 \end{cases}$$

This problem has a unique solution

$$u_n(x,y) = \frac{1}{n^2} \sinh nx \sin ny.$$

We observe that as $n \to \infty$.

$u_n(x,y)$ still oscillates exponentially.

However, the solution of the limiting equation is zero.

Small changes in the initial data.

→ big change in the solution.

∴ It is an ill – posedness problem.

Later on we will see that the Cauchy problem is well – posedness for the wave equation (hyperbolic equation).

Ex：

Consider the diffusion equation, $\begin{cases} u_t = u_{xx} \\ u(x,0) = f(x) \end{cases}$，

$t > 0$， this initial problem is well – posed.

However, the problem $\begin{cases} u_t = u_{xx} \\ u(x,0) = f(x) \end{cases}$， $t < 0$

is ill – posed since the diffusion process can not go "backward".

☞ For the diffusion and wave equations, the initial value problem and initial – boundary value problems are well – posed.
We will discuss them as we learn how to solve them.

# 1-6 Types of second order equations

In this section we will show that the Laplace, wave and heat equations are typical among all second order PDEs. In other words, second order PDEs can be classified into three categories.

☞ Model problem

$$au_{xx} + 2bu_{xy} + cu_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

— a linear equation of second order in two variables.

☞ Theorem

By a linear transformation of the independent variables, the equation can be reduced to one of three forms, as follows

i) Elliptic case： if $b^2 < ac$ , it is reducible to

$u_{xx} + u_{yy} + \cdots = 0.$ (Laplace – like equation)

ii) Hyperbolic case： if $b^2 > ac$ , it is reducible to

$u_{xx} - u_{yy} + \cdots = 0.$ (wave – like equation)

iii) Parabolic case： if $b^2 = ac$ , it is reducible to

$u_{xx} + \cdots = 0.$ (heat – like equation) pf：

i) $b^2 < ac \quad \therefore a \neq 0$

$$a\left(u_{xx} + 2\frac{b}{a}u_{xy} + \frac{c}{a}u_{yy} + \frac{a_1}{a}u_x + \frac{a_2}{a}u_y + \frac{a_0}{a}u\right) = 0$$

$$= a\left(\partial_x{}^2 u + 2\frac{b}{a}\partial_x\partial_y u + \frac{c}{a}\partial_y{}^2 u + lower\ order\right) = 0$$

$\therefore$ We can just consider ( ).

$$\left(\partial_x{}^2 u + 2\frac{b}{a}\partial_x\partial_y u + \frac{c}{a}\partial_y{}^2 u\right)$$

$$= \left(\partial_x + \frac{b}{a}\partial_y\right)^2 u + \left(\frac{c}{a} - \frac{b^2}{a^2}\right)\partial_y{}^2 u$$

$$= \left(\partial_x + \frac{b}{a}\partial_y\right)^2 u + \left(\frac{ac - b^2}{a^2}\right)\partial_y{}^2 u$$

If we let $\left(\partial_x + \frac{b}{a}\partial_y\right)u = \partial_\xi u$ , $\sqrt{\frac{ac-b^2}{a^2}}\partial_y u = \partial_\eta u$ ,

then the equation becomes ${\partial_\xi}^2 u + {\partial_\eta}^2 u = 0$ .

What are $\xi$ and $\eta$ ?

Let $r = \sqrt{\frac{ac-b^2}{a^2}}$.

$$\partial_\xi u = \partial_\xi u(x, y) = \partial_x u + \frac{b}{a}\partial_y u = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi}$$

$\therefore \frac{\partial x}{\partial \xi} = 1$ , $\frac{\partial y}{\partial \xi} = \frac{b}{a}$

$$\partial_\eta u = \partial_\eta u(x,y) = r\partial_y u = \partial_x u \frac{\partial x}{\partial \eta} + \partial_y u \frac{\partial y}{\partial \eta}$$

$$\therefore \frac{\partial x}{\partial \eta} = 0\,,\ \frac{\partial y}{\partial \eta} = r \quad \Rightarrow \quad y = r\eta + f(\xi)$$

$$\therefore x = \xi\,,\ y = r\eta + \frac{b}{a}\xi$$

Then the equation for $\xi$ and $\eta$ is $u_{\xi\xi} + u_{\eta\eta} = 0$.

The proof is similar for other cases.

Ex：

$u_{xx} - 5u_{xy} = 0$ — hyperbolic

$4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0$ — parabolic

$4u_{xx} + 6u_{xy} + 9u_{yy} = 0$ — elliptic

Ex： $u_{xx} - 5u_{xy} = 0$

sol： $\partial_x{}^2 u - 5\partial_x\partial_y u = \left(\partial_x - \frac{5}{2}\partial_y\right)^2 u - \frac{25}{4}u_{yy} = 0$

$\therefore$ We hope $\left(\partial_x - \frac{5}{2}\partial_y\right)u = u_\xi$ , $\frac{5}{2}u_y = u_\eta$ — ①

Let $= x(\xi,\eta)$ , $y = y(\xi,\eta)$ , then $\begin{cases} u_\xi = u_x x_\xi + u_y y_\xi \\ u_\eta = u_x x_\eta + u_y y_\eta \end{cases}$

Compared to ①, we set

$$x_\xi = 1 , y_\xi = -\frac{5}{2} \qquad x_\eta = 0 , y_\eta = \frac{5}{2}$$

$$\Rightarrow x = \xi , y = -\frac{5}{2}\xi + \frac{5}{2}\eta$$

By this transformation, the equation for $\xi$ & $\eta$ is

$$u_{\xi\xi} - u_{\eta\eta} = 0 .$$

verify：

Let $\begin{cases} x = \xi \\ y = -\frac{5}{2}\xi + \frac{5}{2}\eta \end{cases}$, then

$$u_\xi = u_x \cdot x_\xi + u_y \cdot y_\xi = u_x \cdot 1 + u_y \cdot \left(-\frac{5}{2}\right)$$

$$= u_x - \frac{5}{2}u_y$$

$$u_{\xi\xi} = u_{xx} \cdot x_\xi + u_{xy} \cdot y_\xi - \frac{5}{2}u_{yx} \cdot x_\xi - \frac{5}{2}u_{yy} \cdot y_\xi$$

$$= u_{xx} - 5u_{xy} + \frac{25}{4}u_{yy}$$

$$u_\eta = u_x \cdot x_\eta + u_y \cdot y_\eta = \frac{5}{2} u_y$$

$$u_{\eta\eta} = \frac{5}{2} u_{yx} \cdot x_\eta + \frac{5}{2} u_{yy} \cdot y_\eta = \frac{25}{4} u_{yy}$$

$$\therefore\ u_{\xi\xi} - u_{\eta\eta} = u_{xx} - 5u_{xy} = 0$$

$\ast$ Working problem

$$4u_{xx} + 6u_{xy} + 9u_{yy} = 0$$

Remark：

For functions of several variables $u(x, y, z, \cdots)$, one can consult the book.