Chapter 1
Partial differential equations

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1-1 What is a partial differential equation？

Recall that an ODE is an equation which relates the function and it’s ordinary derivatives.

In this case, there is only one independent variable, for example $\frac{dx(t)}{dt} = f(x,t)$.

As for PDE, there is more than one independent variable x, y, z, … .

- u(x, y, z, …)：u is a dependent variable which is an unknown function of u.

i.e. $F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0$

$\Rightarrow$ First order PDE in two independent variables.

Ex：

$$u_x + u_y = 0 \qquad u_t - u_{xx} - u_{yy} = 0$$

$$u_x + uu_y = 0 \qquad u_{xx} - u_{yy} = 0$$

$$u_{tt} - u_{xx} = u^3 \qquad u_{xx} + u_{yy} = 0$$

- A solution of a PDE is a function u(x, y, …) that satisfies the equation is some region of the x, y, … , variables.

Good in PDE： given an equation, and initial condition or boundary condition, try to find the function u(x, y, …).

- In order to classify PDEs, we need to introduce a more general way to write PDEs. i.e. We try to write a PDE as Lu=0 or Lu=g where L is an operator.

Ex： $u_t + u_x = 0 = Lu$

What is Lu？

$Lu = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}$, it’s again a function.

L is a differential operator $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$ acting on functions whose first derivatives exist and are continuous.

Def： L is a linear operator if

L(u+v)=Lu+Lv

L(cu)=cLu

Advantage of linear equation

If Lu=0, Lv=0 $\Rightarrow$ L(u+v)=0.

From two solutions, we produce another solution.

If u and v are both solutions, so is (u+v).

If are all solutions, so is any linear combination

$\Leftrightarrow$ (Superposition principle)

Def：

Homogeneous linear equation： L is linear and Lu=0.

Inhomogeneous linear equation： Lu=g where g is a given function of the independent variables.

Ex：

	$u_x + u_y = 0$	transport
(linear→)	$u_x + yu_y = 0$	transport
	$u_x + uu_y = 0$	shock wave
	$u_{xx} + u_{yy} = 0$	Laplace equation
(nonlinear→)	$u_{xx} - u_{yy} + u^3 = 0$	wave with interaction
	$u_x + u_y = x^2$	

Ex： Which of the following operators are linear？

a) $Lu = u_x + xu_y$

b) $Lu = u_x + uu_y$

c) $Lu = u_x + u_y{}^2$

d) $Lu = u_x + u_y + 1$

e) $Lu = \sqrt{1+x^2}(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$

Some simple examples

- Find u(x,y) such that $u_{xx} = 0$

  $\Rightarrow \quad u_x = c \quad \Rightarrow \quad u_x(x,y) = f(y)$ $\Rightarrow$ u(x,y)=f(y)x+g(y)

- Find u(x,y) such that $u_{xx} + u = 0$

  $\Rightarrow \quad u = f(y)\cos x + g(y)\sin x$

- Find u(x,y) such that $u_{xy} = 0$

  $\Rightarrow \quad u(x,y) = F(y) + G(x)$

Ex： State the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous for the following equations.

a) $u_t - u_{xx} + 1 = 0$

2nd order linear inhomogeneous

$$Lu = u_t - u_{xx}$$

b) $u_t - u_{xxt} + uu_x = 0$

3rd order nonlinear equation

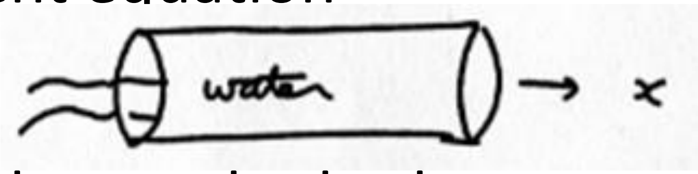
$$Lu = -1 = g$$

1-2 First - order linear equations

In this section, we will start solving some simple partial differential equations.

＊ The constant coefficient equation

Ex： Simple transport



Consider water flowing at a constant rate c

Suppose a pollutant is suspended in the water and let u(x,t) be its concentration in grams/centimeter at time t.

Then the equation for u is given by $u_t + cu_x = 0$.

(transport equation)

Q1： How to derive the equation？

(Suppose diffusion is not considered)

$$\mathrm{M} = \int_0^b u(x,t)dx = \int_{ch}^{b+ch} u(x,t+h)dx$$

(total amount of pollutant in [0,b] at the time t)

To get a differential equation, we differentiate the integral equation with respect to b, and get u(b,t)=u(b+ch,t+h).

Differentiate once more with respect to h, we obtain

$$0 = c \cdot u_x(b+ch,t+h) + u_t(b+ch,t+h) .$$

Set h=0, then the equation becomes $0 = u_t + cu_x$.

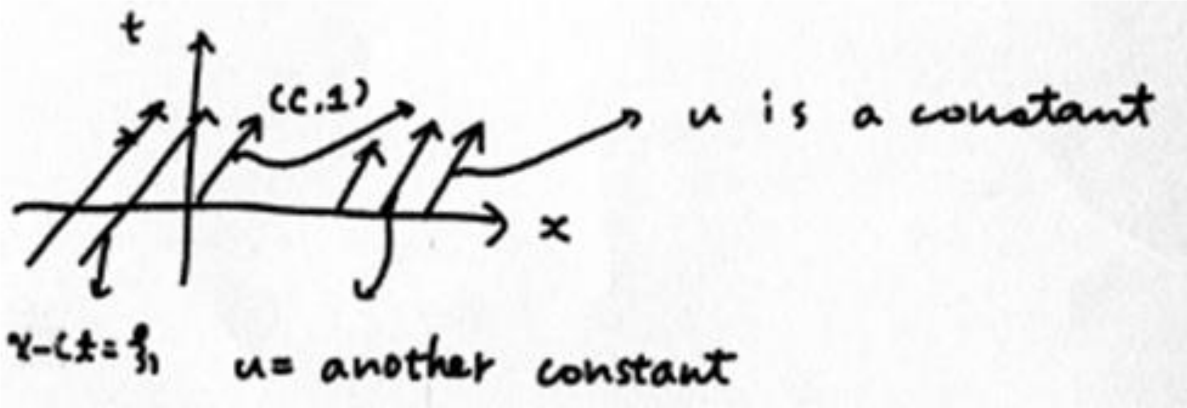
--- transport equation

Q2： How to solve this equation？

We write the equation in the inner product form

$(u_t, u_x)\cdot(1,c)=0$, v=(1,c)

∴ ∇u・v=0 ⇒ u is constant along the v direction.



∴ We know u is a function of the variable x-ct.

i.e. u(x,t) = f(x-ct).

Here f is not yet determined.

Now let us verify if $u_t + cu_x = 0$,

$u_t = -cf'(x\text{-}ct) \qquad u_x = f'(x\text{-}ct) \qquad \therefore \quad u_t + cu_x = 0$

As long as we know the values of u(x,t) at t=0, then u(x,t) is determined at later time.

For example, $\begin{cases} u_t + cu_x = 0 \\ u(x,0) = u_0(x) \end{cases}$ (Initial value problem)

⇒ We know u(x,t) = f(x-ct) for some f.

To find f, we use the initial condition, so we have

$u(x,0) = u_0(x) = f(x)$.

This implies that the solution of the problem is given by

$u_0(x - ct)$.

∗ The constant coefficient equation

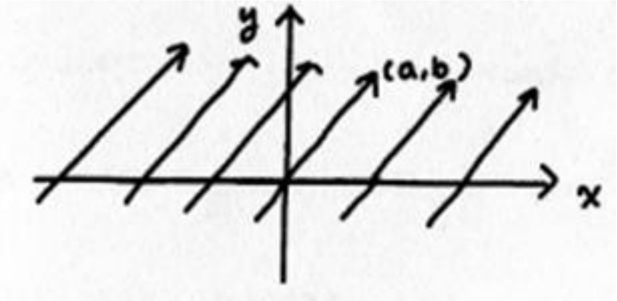
$$au_x + bu_y = 0$$

As before, we can write the equation as

$(a,b)\cdot(u_x,u_y) = 0 = \nabla u \cdot v$ here v=(a,b).

= directional derivative of u along the v-direction.

∴ u is constant in the direction of v.



These lines bx-ay=c are called the **characteristic lines**.

∴ The solution u(x,y)=f(bx-ay)=f(c) for some function f of one variable.

On different characteristic lines, we have different values of c.

Ex： Solve the PDE $\begin{cases} 4u_x - 3u_y = 0 \\ u(0, y) = y^3 \end{cases}$

Q： What is the importance of characteristic lines？

They reduce the PDE to an ODE.

Along the characteristic lines, it becomes an ODE.

Later on, we will elaborate more on the characteristic lines.

∗ The variable coefficient equation

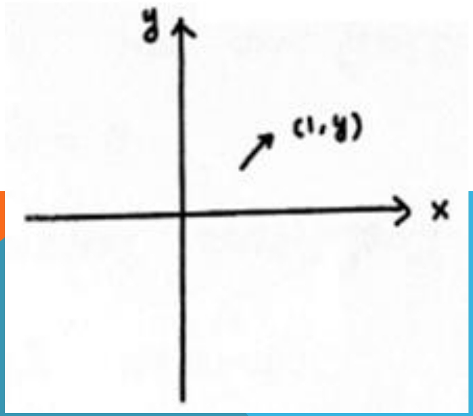
〈 The coefficient is no longer a constant 〉

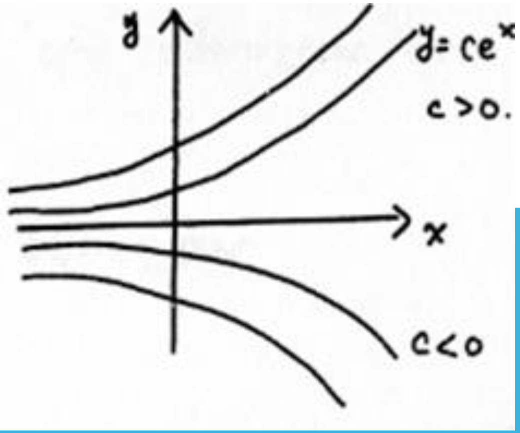
We consider the equation $u_x + yu_y = 0$

We can still write it as $(u_x, u_y) \cdot (1, y) = 0$.

$\Rightarrow \nabla u \cdot (1, y) = 0$

∴ u is constant along the direction (1,y).





Let us find the characteristic curves.

Since the slope of the characteristic curves is $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{1}$.

$\therefore$ $\mathrm{y} = ce^{x}$, $\therefore$ u is a constant along the curves
$\mathrm{y} = \mathrm{c}e^{x}$.

Given different c, we have different values of u,

$\therefore$ u depends on c only.

$\Rightarrow$ u is a function of one variable c.

$\Rightarrow \mathrm{u} = \mathrm{f(c)} = \mathrm{f}(ye^{-x})$.

$\Rightarrow$ This is the general solution to the PDE
$\mathrm{u}_x + \mathrm{yu}_y = 0$.

- To verify that it indeed solves the PDE, we compute

$$u_x + yu_y = -ye^{-x} \cdot f' + ye^{-x} \cdot f' y = 0$$

$\therefore$ To make the solution unique, we only need to impose one more condition to determine f.

Ex： Solve $\begin{cases} u_x + yu_y = 0 \\ u(0, y) = y^3 \end{cases}$

sol：

We already have the general solution $u = f(ye^{-x})$

Now the initial condition gives us

$f(y) = y^3 \Rightarrow u = y^3 e^{-3x}$.

Ex： Solve the PDE $u_x + 2xy^2 u_y = 0$.

sol： $(u_x, u_y) \cdot (1, 2xy^2) = 0$

$\therefore$ u is constant along the characteristic curves whose slope is $\frac{dy}{dx} = \frac{2xy^2}{1}$.

To solve this ODE, we use the separation of variables,

$$\frac{dy}{dy^2} = 2x dx \Rightarrow \frac{-1}{y} = x^2 + c$$

$\therefore y = \frac{-1}{x^2+c}$

$\therefore$ Different value of c gives different value of u.

$\therefore u = f(-\frac{1}{y} - x^2)$.

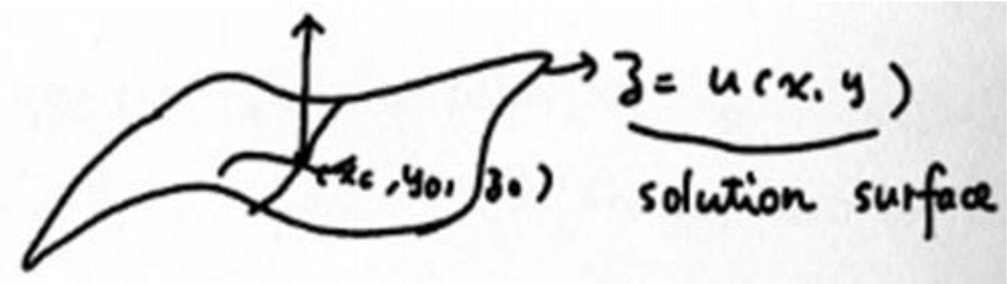
You can verify the solution by direct differentiation.

Q1： What kind of conditions can we impose to determine the function f？

Q2： How to solve $u_x + yu_y = c$ ？

☞ How to solve where a, b, c are constants ？

Let z=u(x,y), it is a surface in $\mathcal{R}^3$.



at the point (x_0, y_0, z_0), the normal is $\left(u_x, u_y, -1\right) = n$

$\therefore$ The equation $au_x + bu_y = c$ can be rewritten as $(a, b, c) \cdot n = 0$

$$\begin{cases} \dfrac{dx}{dt} = a & x(0) = s \\ \dfrac{dy}{dt} = b & y(0) = 0 \\ \dfrac{dz}{dt} = c & z(0) = h(s) \end{cases}$$

$\therefore$ x=at+s, y=bt, z=ct+h(s)

eliminate s and t, $\frac{b}{a}x - y = \frac{b}{a}s$

$\therefore \mathrm{s} = \mathrm{x} - \frac{a}{b}\mathrm{y}, \mathrm{t} = \frac{y}{b}$

$\Rightarrow \mathrm{z} = \frac{c}{b}y + h\left(\mathrm{x} - \frac{a}{b}\mathrm{y}\right) = u(x, y)$

Compute $\mathrm{u}_x = h', \mathrm{u}_y = \frac{c}{b} - \frac{a}{b}h'$

$\mathrm{au}_x + \mathrm{bu}_y = ah' + c - ah' = c$

Q： How to deal with the problem

$\mathrm{a(x, y, u)u}_x + \mathrm{b(x, y, u)u}_y = c\mathrm{(x, y, u)}$?

1-3 Flows, vibrations and diffusions

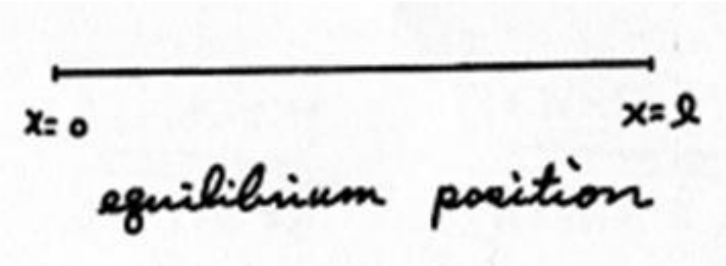
In this section, we will introduce some equations from physics.

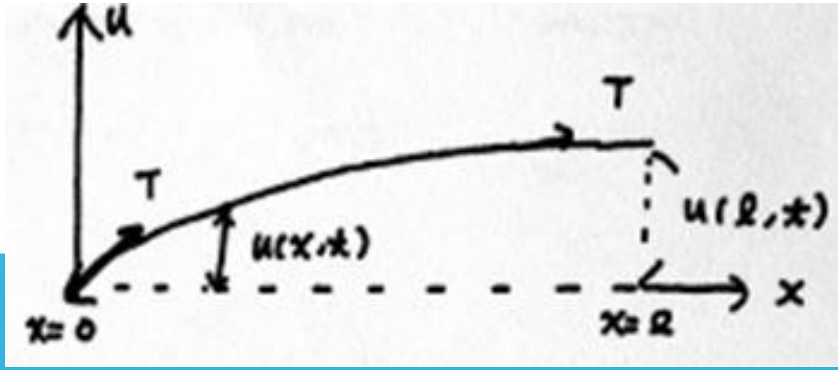
In the previous note, we have already desired the transport equation.

Now we will look at the others.

∗ Vibrating string

Consider a flexible, elastic homogeneous sting (constant density ρ) of length l. (like a guitar string) Let u(x,t) be its displacement from equilibrium position at time t and position x.





Since the string is flexible, the force T(x,t) is along the tangential direction.

The slope of the tangent at (x_1,t) is $u_x(x_1,t)$.

$\therefore$ The horizontal force is $\dfrac{T}{\sqrt{u_x^{\,2}+1}}$ which has no effect on the displacement u.

The vertical force is $\dfrac{Tu_x}{\sqrt{u_x^{\,2}+1}}$.

By using the Newton's law F=ma, we have

$$\frac{Tu_x(x_1,t)}{\sqrt{u_x^{\,2}+1}}-\frac{Tu_x(x_0,t)}{\sqrt{u_x^{\,2}+1}}=\int_{x_0}^{x_1}\rho u_{tt}\,dx \quad .$$

Q： How to get an equation for u？

$$\frac{Tu_x(x_1,t)}{\sqrt{1+u_x{}^2(x_1,t)}} - \frac{Tu_x(x_0,t)}{\sqrt{1+u_x{}^2(x_0,t)}} = \int_{x_0}^{x_1} \rho u_{tt}\, dx$$

To get an equation for u, we notice that the motion is small, so u_x is small.

Therefore $1 + u_x{}^2 \approx 1$, then if we differentiate the equation with respect to x_1, we get

$$\mathrm{T}u_{xx} = \rho u_{tt} \quad \Rightarrow \quad u_{tt} = c^2 u_{xx} \quad \text{where} \quad c = \sqrt{\frac{T}{\rho}} \ .$$

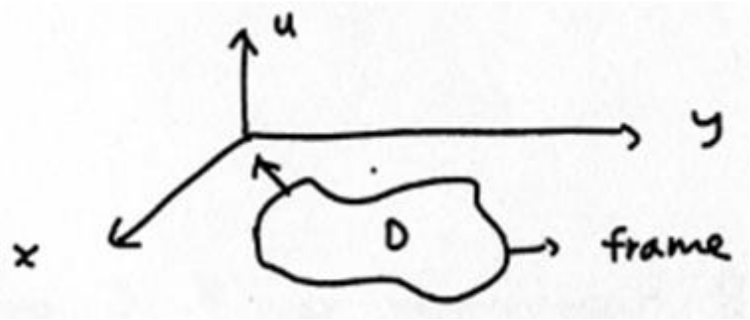
This is the famous wave equation with the wave speed c.

There are some other variations of the wave equations, one can consult the book.

Ex： Vibrating drumhead

Now we try to generalize the previous derivation to 2 – dimensions.

Suppose we have a membrane stretched over a frame in the xy plane.

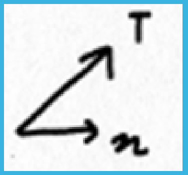


Let u(x,y,t) denote the vertical displacement and there is no horizontal motion.

∂D → the boundary of D.

How to apply the Newton's law？ $F = ma = \int_D \rho u_{tt} \, dxdy$

What is the force acting on ∂D？ $F = \int_{\partial D} T \frac{\partial u}{\partial n} \, ds$



Ex： diffusion

If a dye is diffusing through the liquid, then it will move from 高密度 to 低密度.

Let u(x,t) be the concentration of the dye at position x of the pipe at time t.

According to Fick's law of diffusion：the rate of motion is proportional to the concentration gradient.

The mass of dye between x_0 and x_1 is $M(t) = \int_{x_0}^{x_1} u(x,t)\,dx$,

then $$\frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x,t)\,dx$$.

By Fick's law, $\frac{dM}{dt} = ku_x(x_1,t) - ku_x(x_0,t)$.

Therefore, one has $\int_{x_0}^{x_1} u_t dx = ku_x(x_1,t) - ku_x(x_0,t)$.

To get a differential equation, we differentiate it with respect to x_1 and obtain $u_t = ku_{xx}$.

< Diffusion equation (heat equation) >

In higher space dimensions, the diffusion equation can be written as $u_t = k\Delta u$.